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Bulk scaling limits for random normal matrix ensembles near singularities

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Abstract

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In this thesis, we study the scaling limit of eigenvalues of a random normal matrix model at a singularity. The eigenvalues of a random normal matrix follow the Boltzmann-Gibbs distribution with an external potential. With a suitable growth condition of the potential, the eigenvalues are asymptotically distributed according to the equilibrium measure with a compact support as the size of the matrix goes to infinity. We consider two types of singularities: a bulk singularity, a point in the interior of the spectrum at which the equilibrium density vanishes, and a conical singularity, a logarithmic singularity in the interior of the spectrum. We prove some universality results for scaling limits which show a dominant part in the Taylor expansion of the potential determines the microscopic properties near a singularity.

Key words: Random normal matrix, Bulk singularity, Conical singularity, Ward's equation, Universality
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Chapter 1

Introduction

Random matrix theory has developed with applications in various areas of mathematics and physics. Random matrices were first considered by Wishart [42] in the context of multivariate statistics to study the sample covariance matrices. In physics, random matrix models were introduced by Wigner [41] to understand the excitation spectra of heavy nuclei. He described the statistical properties of a heavy nuclei through a large matrix with random entries. Random matrix ensembles were established mathematically in Dyson's papers [15, 16, 17], where the orthogonal, unitary, and symplectic ensembles were studied. Dyson also found the relation between random matrix ensembles and one dimensional integrable systems. By calculating the probability distribution of eigenvalues of the three ensembles, an exact mathematical correspondence was revealed between the eigenvalue distributions and the statistics of a one-dimensional Coulomb gas at three special temperatures. Following these studies, there have been many remarkable advances in various models of random matrix theory including random symmetric, Hermitian, and non-Hermitian matrices.

1.1 Random normal matrix models

In this thesis, we study the distribution of eigenvalues of the random normal matrix model. Random normal matrices were introduced by Chau and Yu [13] in connection with the Quantum Hall Effect in physics. Chau and Zaboronsky [14] studied the structure of random normal matrix model and derived the relation between random normal matrix and the Toda lattice hierarchy. Wiegmann and Zabrodin [38, 39] found the relation between the density of eigenvalues in the random normal matrix model and conformal maps of simply connected domains with an analytic boundary to a unit disk, and Elbau and Felder [18] provided a mathematically rigorous proof by means of polynomial curves.

In the random normal matrix model with an external potential function $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$, we study normal matrices M, i.e. $MM^* = M^*M$, of size n which are picked randomly with respect to a probability measure of the form

$$d\mathcal{P}_n(M) = \frac{1}{\mathcal{Z}_n} e^{-n\,\mathrm{tr}\,Q(M)} dM. \tag{1.1.1}$$

Here, dM is the surface measure on the space of normal $n \times n$ matrices induced by the Euclidean metric on all complex matrices, $\mathcal{Z}_n$ is a normalizing factor, and $\mathrm{tr}\,Q(M)$ is the trace of the matrix $Q(M)$ given by $\sum_{j=1}^{n} Q(\lambda_j)$ with the eigenvalues λ_j of M. Assuming that Q has sufficient growth near infinity, the normalizing factor $\mathcal{Z}_n$ is well-defined.

Under the influence of the potential Q, the eigenvalues tend to occupy a compact set, called the *droplet*, as the size of matrix goes to infinity. In particular, at the macroscopic level, the eigenvalue density is asymptotically equal to the Laplacian of the potential Q and the droplet can be expressed through a solution of an inverse problem related to Hele-Shaw flows. This macroscopic property of eigenvalues was investigated by Wiegmann-Zabrodin [40], Etingof-Ma [19], and Hedenmalm-Makarov [25].

An important feature of random normal matrix model is that we can compute the exact joint probability distribution for the eigenvalues. The joint probability distribution follows the Boltzmann-Gibbs law,

$$d\mathbf{P}_n(\mathbf{z}) = \frac{1}{Z_n} e^{-H_n(\mathbf{z})} dA^{\otimes n}(\mathbf{z}), \quad \mathbf{z} = (z_j)_{j=1}^{n} \in \mathbb{C}^n, \tag{1.1.2}$$

where H_n is the energy for a two-dimensional Coulomb gas model,

$$H_n(\mathbf{z}) = -\sum_{j \neq k} \log|z_j - z_k| + n \sum_{j=1}^{n} Q(z_j),$$

$dA^{\otimes n}$ is Lebesgue measure in $\mathbb{C}^n$ divided by π^n, and $Z_n = \int_{\mathbb{C}^n} e^{-H_n} dA^{\otimes n}$ is the normalizing constant. This eigenvalues system can be interpreted as a special case of the two-dimensional one-component plasma at a special temperature. Another important feature of this system is that it forms a determinantal point process, while the two-dimensional one-component plasma does not necessarily have a determinantal structure.

1.2 Microscopic properties of random normal matrix ensembles

This thesis focuses on the microscopic property of eigenvalues, one of the main interests in random matrix theory. A distinguishing feature, which makes the study of microscopic properties so interesting, is "universality": the microscopic statistics of eigenvalues does not depend on the details of the system. Our ultimate goal of this study is thus to prove the universality of random normal matrix ensembles.

The microscopic behavior of eigenvalues is studied by rescaling them at points both in the interior and on the boundary of the droplet. Let $\{\zeta_j\}_1^n$ be the system of eigenvalues of the random normal matrix model (1.1.2). We define a rescaled eigenvalue system $\{z_j\}_1^n$ at a fixed point p on a appropriate microscopic scale $r_n = r_n(p)$ with $r_n \to 0$ as $n \to \infty$ by letting $z_j = r_n^{-1}(\zeta_j - p)$ for $j = 1, \cdots, n$. The purpose of the study is to obtain the limit of this rescaled system as $n \to \infty$.

The simplest example is the case when $Q(z) = |z|^2$, called the Ginibre ensemble, which is introduced by Ginibre [23]. In this case, the droplet is the unit disk and for a point p in the interior of the droplet, the rescaled system $\{z_j\}$ defined by $z_j = \sqrt{n}\,(\zeta_j - p)$ converges to the determinantal point process with Ginibre kernel $G(z,w) = e^{z\bar{w}-|z|^2/2-|w|^2/2}$. At a point p on the boundary of the droplet, the rescaled system $\{z_j\}$ defined by $z_j = e^{-i\theta}\sqrt{n}(\zeta_j - p)$ where $e^{i\theta}$ is the outer normal at p converges to the determinantal process with kernel $\frac{1}{2}\operatorname{erfc}\left((z+\bar{w})/\sqrt{2}\right) G(z,w)$. Details of related studies can be found in Forrester [20, 21], Borodin-Sinclair [11], and Ameur-Kang-Makarov [5].

For general potential Q, Ameur, Hedenmalm, and Makarov proved the universality of the bulk scaling limit in [2], and Ameur, Kang, and Makarov proved the universality of the edge scaling limit for radially symmetric potential Q in [5]. Both universality results were proved with the microscopic scale $r_n = (n\Delta Q(p))^{-1/2}$ under the assumption of $\Delta Q(p) > 0$.

1.3 Rescaled eigenvalue system near a singularity

We study the microscopic property of eigenvalues near a singularity in the interior of the droplet. We consider two types of singularities:

- Bulk singularity: an isolated point in the interior of the droplet at which ΔQ vanishes.
- Conical singularity: a logarithmic singularity in the interior of the droplet obtained by the perturbation of the potential,

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n} \log|\zeta - p|\,, \quad c > -1.$$

Bulk singularities

For an integer $d > 1$, we say Q has a *bulk singularity of type* $2d-2$ at p if p is in the interior of the droplet and the Taylor expansion of ΔQ at p is of the form

$$\Delta Q(\zeta) = P(\zeta - p) + O(|\zeta - p|^{2d-1}), \quad \zeta \to p,$$

where $P(x+iy)$ is a homogeneous polynomial in x, y of degree $2d-2$. The Taylor expansion of Q gives the canonical decomposition

$$Q = Q_0(\zeta - p) + h(\zeta - p) + O(|\zeta - p|^{2d+1}), \quad \zeta \to p,$$

where Q_0 is the dominant part, which is homogeneous of degree $2d$, and h is a harmonic function.

In general a bulk singularity tends to repel eigenvalues away. A proper microscopic scale r_n at a bulk singularity p is chosen to separate the other eigenvalues from p, which is determined by the degree of the dominant term in the Taylor series of ΔQ at p. The microscopic scale r_n at a bulk singularity of type $2d-2$ is $O(n^{-1/2d})$, which is relatively coarse in comparison with the one in the regular case: $r_n(p) = O(n^{-1/2})$ when $\Delta Q(p) > 0$.

Let $\{\zeta_j\}_1^n$ be the eigenvalue system of the random normal matrix model associated with Q and $\{z_j\}_1^n$ be the rescaled eigenvalue system defined by $z_j = r_n^{-1}(\zeta_j - p)$. The limiting distribution of the rescaled system can be described by the reproducing kernel of a Bergman space, which consists of entire functions in $L^2(\mu_0)$ where $d\mu_0 = e^{-Q_0}\, dA$ and Q_0 is the dominant part of Q.

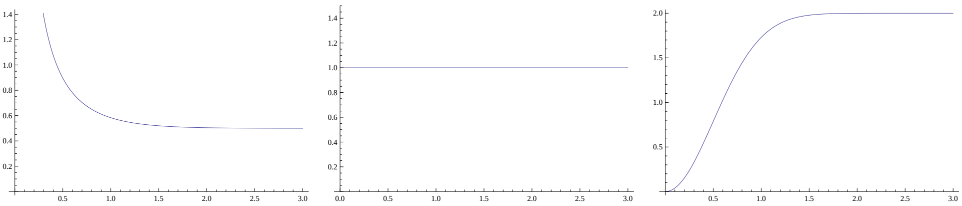

Figure 1.1: Microscopic densities $R(x)$ where x is on the positive real line for $V_n(z) = \frac{1}{2}|z|^2 + \frac{1}{n}\log|z|$, $Q(z) = |z|^2$, and $V_n(z) = 2|z|^2 - \frac{2}{n}\log|z|$.

In Chapter 3, we prove the universality of scaling limits at a bulk singularity for the cases when Q is radially symmetric and when Q is a homogeneous polynomial.

Conical singularities

For a fixed real parameter $c > -1$, we consider the n-dependent potential

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n}\log|\zeta - p|$$

and an eigenvalue system $\{\zeta_j\}_1^n$ of the random normal matrix model associated with the potential V_n. We say V_n has a *conical singularity of order c* at p. This conical singularity appears in the study of Quantum Hall states on singular surfaces [12], especially in relation with QH state on a cone. As n goes to infinity, the effect of the perturbation in the potential tends to disappear in the macroscopic view. On the microscopic level, a conical singularity significantly affects the density of eigenvalues (See Figure 1.1). If c is negative, then eigenvalues are distributed more densely near the singularity compared to the original case. On the other hand, if c is positive, then the microscopic density vanishes at the singularity.

A description of the microscopic property of the rescaled eigenvalue system is given in terms of the reproducing kernel for some generalized Fock spaces of entire functions. Assume that $p = 0$ and $Q(x + iy)$ is a homogeneous polynomial in x, y of degree $2d$. Let Q_0 be the dominant part of Q,

$$Q_0(\zeta) = Q(\zeta) - 2\,\mathrm{Re}\left(\frac{\partial^{2d}Q(0)}{(2d)!}\zeta^{2d}\right).$$

With a microscopic scale $r_n = O\left(n^{-1/2d}\right)$, a rescaled system $\{z_j\}_1^n$ defined by $z_j = r_n^{-1}\zeta_j$ has a nontrivial scaling limit. Write $V_0(z) = Q_0(z) - 2c\log|z|$ and $d\mu_0 = e^{-V_0}dA$. The microscopic density of this rescaled system $\{z_j\}_1^n$ is obtained by

$$R(z) = L_0(z,z)e^{-V_0(z)}$$

where L_0 is the reproducing kernel of the space which consists of entire functions in $L^2(\mu_0)$. We prove existence and universality of the scaling limit near the conical singularity in Chapter 4.

Notations

∘ We write $\partial = (\partial_x - i\partial_y)/2$ and $\bar\partial = (\partial_x + i\partial_y)/2$ for the complex derivatives and $\Delta = \partial\bar\partial$ for 1/4 times the standard Laplacian on $\mathbb{C}$.
∘ $D(p,r)$ is the open disk centered at p with radius r.
∘ $dA = dxdy/\pi$ is the Lebesgue measure divided by π.
∘ A continuous function $h(z,w)$ is called Hermitian if $h(z,w) = \overline{h(w,z)}$. h is called Hermitian-analytic (Hermitian-entire) if h is analytic (entire) in z and $\bar w$.
∘ The indicator function of a set S is denoted by χ_S.
∘ Pol(n) is the linear space of analytic polynomials of degree less than n.

Chapter 2

Preliminaries

In this chapter, we present a brief survey on random normal matrices as preliminaries. It provides the theoretical background and notations for the thesis.

2.1 Random normal matrix ensembles

Let $\mathcal{N}_n$ be the space of all $n \times n$ normal matrices M. We study the random normal matrix ensemble defined by the probability distribution

$$d\mathcal{P}_n(M) = \frac{1}{\mathcal{Z}_n} e^{-n \operatorname{tr} Q(M)} dM. \tag{2.1.1}$$

Here, the measure dM on the space $\mathcal{N}_n$ is the Riemannian volume form induced by the flat metric of $\mathbb{C}^{n^2}$. The normalizing constant $\mathcal{Z}_n$ is defined as an integral

$$\mathcal{Z}_n = \int_{\mathcal{N}_n} e^{-n \operatorname{tr} Q(M)} dM,$$

which is called the *partition function* of the random normal matrix ensemble. Basically, in this thesis, the function $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is assumed to be lower semi-continuous satisfying the growth condition

$$\liminf_{|\zeta| \to \infty} \frac{Q(\zeta)}{2 \log |\zeta|} > 1. \tag{2.1.2}$$

The integral $\mathcal{Z}_n$ is convergent with this growth condition, so that the probability distribution $\mathcal{P}_n$ is well-defined. We call Q the *potential* of the random normal matrix ensemble. For each $M \in \mathcal{N}_n$, $\operatorname{tr} Q(M)$ is the trace of the matrix $Q(M)$ and it is equal to $\sum_{j=1}^{n} Q(\zeta_j)$ where ζ_j are the eigenvalues of M. We call the space $\mathcal{N}_n$ equipped with the measure (2.1.1) the *random normal matrix ensemble associated with* Q.

2.1.1 The joint distribution of eigenvalues

From (2.1.1), one can derive the joint distribution of eigenvalues of the random normal matrix ensemble. It is well-known that the joint distribution of the eigenvalues $\{\zeta_j\}_{j=1}^{n}$ is of the form for $\boldsymbol{\zeta} = (\zeta_j)_{j=1}^{n} \in \mathbb{C}^n$,

$$d\mathbf{P}_n(\boldsymbol{\zeta}) = \frac{1}{Z_n} \prod_{1\le j<k\le n} |\zeta_j - \zeta_k|^2 \, e^{-n\sum_{j=1}^{n} Q(\zeta_j)} dA^{\otimes n}(\boldsymbol{\zeta}), \tag{2.1.3}$$

where $dA^{\otimes n}(\boldsymbol{\zeta}) = dA(\zeta_1)\cdots dA(\zeta_n)$ with $dA(\zeta_j) = d^2\zeta_j/\pi$ and

$$Z_n := \int_{\mathbb{C}^n} \prod_{1\le j<k\le n} |\zeta_j - \zeta_k|^2 \, e^{-\sum_{j=1}^{n} Q(\zeta_j)} dA^{\otimes n}(\boldsymbol{\zeta})$$

is the normalizing constant. From now on, we refer to Z_n as the *partition function* of the random normal matrix ensemble associated with Q. We remark that the joint distribution $\mathbf{P}_n$ vanishes when two eigenvalues are equal. Thus, it can be regarded as a joint distribution on the set

$$\{\boldsymbol{\zeta} = (\zeta_j) \in \mathbb{C}^n \,|\, \zeta_j \neq \zeta_k \text{ for } j \neq k\}.$$

The key idea to obtain the joint density of eigenvalues from (2.1.1) is integrating out the other variables except the eigenvalues, which is well-known argument in the random matrix theory. See [21], [33].

Indeed, a normal matrix M can be diagonalized by a unitary matrix U, $M = UDU^*$, where D is the diagonal matrix with the eigenvalues of M as the diagonal entries. The unitary matrix U can be chosen up to a right multiplication by a diagonal unitary matrix, so we take $U \in U(n)/U(1)^n$. The following proposition gives the factorization of the measure dM on the space $\mathcal{N}_n$.

Proposition 2.1.1. *The measure dM in* (2.1.1) *is given by the formula*

$$dM = dU \prod_{1\le j<k\le n} |\zeta_j - \zeta_k|^2 \prod_{j=1}^{n} d^2\zeta_j,$$

where $M = U\,\mathrm{diag}(\zeta_1, \cdots, \zeta_n)U^*$ *for* $U \in U(n)$ *and* dU *denotes the invariant measure on* $U(n)/U(1)^n$.

For details of the proof, we refer to [18] and [43].

2.1.2 Physical context

The joint distribution (2.1.1) of the eigenvalues can be expressed in the form

$$d\mathbf{P}_n(\mathbf{z}) = \frac{1}{Z_n} e^{-H_n(\mathbf{z})} dA^{\otimes n}(\mathbf{z}), \quad \mathbf{z} = (z_j)_{j=1}^n \in \mathbb{C}^n$$

where H_n is the energy for a two-dimensional Coulomb gas model,

$$H_n(\mathbf{z}) = -\sum_{j \neq k} \log|z_j - z_k| + n \sum_{j=1}^n Q(z_j). \tag{2.1.4}$$

The joint distribution of eigenvalues is equal to the Boltzmann-Gibbs measure for the two-dimensional one-component plasma model, which consists of n identical electrically charged point particles in the complex plane influenced by an external field nQ at a specific temperature. When the particles are located at points $\{z_j\}$, the first term of H_n gives the energy of the mutual repulsion among charges and the second term gives the total potential energy of external filed. While this confining potential keeps the charges at a finite distance, the charges tend to be distributed evenly due to the repulsion among them.

Remark 2.1.2 (β-ensembles). For $\beta > 0$, the Boltzmann-Gibbs measure for the two-dimensional one-component plasma model at the inverse temperature β is given by

$$d\mathbf{P}_{n,\beta}(\mathbf{z}) = \frac{1}{Z_n} e^{-\beta H_n(\mathbf{z})} dA^{\otimes n}(\mathbf{z}), \quad \mathbf{z} = (z_j)_{j=1}^n \in \mathbb{C}^n \tag{2.1.5}$$

where H is the Hamiltonian (2.1.4) of the model. A random normal matrix model is a special case of this model when $\beta = 1$. We call the system of particles distributed according to (2.1.5) the *β-ensemble*. We refer to [5, 25, 35]. In recent papers [9, 10, 30, 31], the central limit theorem for fluctuations and the local densities of β-ensembles were studied.

2.2 Determinantal point processes

The eigenvalues of random normal matrices can be described as a random point process. More precisely, they form a determinantal point process on the complex plane. We start with some basic notions and definitions mainly from [28] to investigate this in detail.

2.2.1 Definitions

Let X be a locally compact topological space equipped with a complete and separable metric and μ be a Radon measure on X. The most common example of (X, μ) is that X is an open subset of $\mathbb{R}^d$ and μ is the d-dimensional Lebesgue measure restricted to X.

A *point process* Θ on X is a random integer-valued positive Radon measure on X. We call Θ *simple* if Θ takes values either 0 or 1 at a single point in X. One way to understand a simple point process is to consider it as a random discrete subset of X. That is, for any Borel set D of X, one can think of $\Theta(D)$ as a random variable which gives the number of points contained in D. Form now on, we only consider simple point processes.

Let $\mathcal{M}(X)$ be the space of σ-finite Borel measures on X. Then, for any Borel sets D_k and non-negative integers n_k, $1 \le k \le m$, we define a subset

$$\{\nu \in \mathcal{M}(X) \ : \nu(D_k) = n_k, \ 1 \le k \le m\}$$

of $\mathcal{M}(X)$ and refer to those subsets *cylinder sets*. We note that for each point process Θ, the probabilities of cylinder sets

$$\mathbf{P}[\Theta(D_k) = n_k, \ 1 \le k \le m]$$

describe the distribution of Θ. In this regard, we define *joint intensities*, or *correlation functions* of a point process as follows:

Definition 2.2.1. Let Θ be a simple point process on X. We call a locally integrable function $\rho_k : X^k \to [0, +\infty)$ the k-point joint intensity (or k-point correlation function) of Θ with respect to μ if for any mutually disjoint subsets $D_1, \dots, D_r$ of X and $k = \sum_{i=1}^r k_i$,

$$\mathbf{E}\left[\prod_{i=1}^{r} \binom{\Theta(D_i)}{k_i} k_i!\right] = \int_{\prod_i D_i^{\otimes k_i}} \rho_k(x_1, \dots, x_k)\, d\mu(x_1) \dots d\mu(x_k).$$

Remark 2.2.2. If X is an open set in $\mathbb{R}^d$ and μ is the Lebesgue measure on $\mathbb{R}^d$, then for distinct $x_1, \dots, x_k$ on X, we have

$$\rho_k(x_1, \dots, x_k) = \lim_{\epsilon \to 0} \frac{\mathbf{P}\left[\Theta(B(x_j, \epsilon)) \geq 1 \text{ for each } 1 \leq j \leq k\right]}{\mathrm{vol}(B(x_1, \epsilon))^k}.$$

Here, $B(x, \epsilon)$ is a ball centered at x with radius ϵ.

We now give the definition of determinantal point processes.

Definition 2.2.3. Let $K(x, y) : X \times X \to \mathbb{C}$ be a measurable function. We say that a simple point process Θ on X is a *determinantal point process* with kernel K if the correlation functions are given by

$$\rho_k(x_1, \dots, x_k) = \det(K(x_i, x_j))_{1 \leq i, j \leq k},$$

for all $k \geq 1$ and $x_1, \dots, x_k \in X$.

We call this kernel K the *correlation kernel* of the process.

2.2.2 Eigenvalue processes of random normal matrices

Let $\Theta_n = \{\zeta_j\}_{j=1}^n$ be the eigenvalues of a $n \times n$ random normal matrix distributed according to the measure (2.1.1). As in section 2.1.1, the joint distribution of eigenvalues is given by the formula (2.1.3). Then, Θ_n forms a simple point process on $\mathbb{C}$ in the sense that for any Borel set D of $\mathbb{C}$, $\Theta_n(D)$ represents the number of eigenvalues which are contained in D. Moreover, it turns out that this process is determinantal as the theorem below.

For given potential Q, let μ_n be the measure $d\mu_n = e^{-nQ}\, dA$ on $\mathbb{C}$ and $L^2(\mu_n)$ be the space of functions on $\mathbb{C}$ equipped with the norm

$$\| f \|_{L^2(\mu_n)}^2 = \int_{\mathbb{C}} |f|^2 \, e^{-nQ} \, dA.$$

We define $\mathcal{P}_n$ to be the subspace of $L^2(\mu_n)$ consisting of holomorphic polynomials of degree less than n.

Theorem 2.2.4. *Let Θ_n be the set of eigenvalues of random normal matrix ensemble associated with the potential Q. Then, Θ_n forms a determinantal point process on $\mathbb{C}$ with kernel*

$$\mathbf{K}_n(\zeta, \eta) = \mathbf{k}_n(\zeta, \eta) \, e^{-n(Q(\zeta)+Q(\eta))/2},$$

where $k_n(\zeta, \eta)$ is the reproducing kernel of the space $\mathcal{P}_n$.

If we denote the k-point joint intensity of the process Θ_n by $\mathbf{R}_{n,k}$, then

$$\mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k) = \det\left(\mathbf{K}_n(\zeta_i, \zeta_j)\right)_{1 \le i,j \le k}$$

and the corresponding correlation kernel $\mathbf{K}_n$ can be written as

$$\mathbf{K}_n(\zeta, \eta) = \sum_{j=0}^{n-1} p_j(\zeta)\overline{p_j(\eta)}\, e^{-n(Q(\zeta)+Q(\eta))/2},$$

where p_j is an orthonormal polynomial of degree j with respect to the measure μ_n. A proof of this theorem, written for the case of Hermitian random matrices, is given in [33]. The theorem for the case of random normal matrices can be proved in the same way.

We here give a well-known example.

Example 2.2.5. (Ginibre ensemble) For the case when $Q(\zeta) = |\zeta|^2$, the eigenvalue system $\{\zeta_j\}_1^n$ of random normal matrix ensembles associated Q is called Ginibre ensemble. The system $\{\zeta_j\}_1^n$ forms a determinantal point process with kernel

$$\mathbf{K}_n(\zeta, \eta) = \sum_{j=0}^{n-1} \frac{n^{j+1}(\zeta\bar{\eta})^j}{j!}\, e^{-n\left(|\zeta|^2+|\eta|^2\right)/2}.$$

It should be pointed out that the distribution of eigenvalues as a point process is determined by its correlation functions $\mathbf{R}_{n,k}$. Thus, we focus on the limit of the (properly rescaled) correlation kernels $\mathbf{K}_n(z, w)$ to understand the limiting behavior of eigenvalues as n tends to infinity.

2.3 Logarithmic potential theory

In this section, we introduce some basic notions and known facts in the logarithmic potential theory in connection with the eigenvalues system. The limiting behavior of eigenvalues is closely related to the minimal energy problem with logarithmic kernel under an external field. More precisely, if the potential (external field) is strong enough near infinity, the eigenvalues tend to accumulate on a compact subset and asymptotically distributed according to the Frostman's equilibrium measure when the size of matrix tends to infinity.

2.3.1 Equilibrium measure and droplet

We here follow the definitions in [34]. First, we start with the definition of admissible functions.

Definition 2.3.1. we say a potential $Q : \mathbb{C} \to (-\infty, \infty]$ is *admissible* if it satisfies the following conditions:

(i) Q is lower semi-continuous,

(ii) The set $\{z \in \mathbb{C} \,|\, Q(z) < \infty\}$ has positive capacity,

(iii) $\lim_{|z|\to\infty}\{Q(z) - 2\log|z|\} = \infty$.

Let $\mathcal{P}_c(\mathbb{C})$ be the collection of all positive, compactly supported Borel probability measures on $\mathbb{C}$. For each $\mu \in \mathcal{P}_c(\mathbb{C})$, we define the weighted energy integral by

$$I_Q(\mu) := \iint \log \frac{1}{|z-t|^2}\, d\mu(z)d\mu(t) + 2\int Q\, d\mu.$$

There exists a measure that minimizes the weighted energy as the following theorem states.

Theorem 2.3.2 (Frostman)**.** *Let Q be an admissible potential. Then there exists a unique probability measure $\sigma = \sigma_Q \in \mathcal{P}_c(\mathbb{C})$ such that*

$$I_Q(\sigma) = \inf\{I_Q(\mu) \,|\, \mu \in \mathcal{P}_c(\mathbb{C})\}.$$

Moreover, σ has finite logarithmic energy and $\operatorname{supp}(\sigma)$ *is compact on* $\mathbb{C}$.

The measure σ is called the *equilibrium measure* associated with Q. We denote the support of σ by $S = S_Q$ called the *droplet*.

For any measure $\mu \in \mathcal{P}_c(\mathbb{C})$, the logarithmic potential of μ is defined by

$$U^\mu(z) = \int_{\mathbb{C}} \log \frac{1}{|z-w|}\, d\mu(w), \quad z \in \mathbb{C}.$$

The following proposition gives a relation between the potential Q and the logarithmic potential of the equilibrium measure σ. Moreover, this relation gives a characterization of the equilibrium measure.

Proposition 2.3.3. *Let σ be the equilibrium measure associated with Q and U^σ be its logarithmic potential. Then the following properties hold.*

(i) $2U^\sigma(z) + Q(z) \geq c_Q$ *for quasi-every $z \in \mathbb{C}$,*

(ii) $2U^\sigma(z) + Q(z) \leq c_Q$ *for all $z \in S$,*

where the constant c_Q is defined by $c_Q := I_Q(\sigma) - \int Q\, d\sigma$.

Conversely, if $\mu \in \mathcal{P}_c(\mathbb{C})$ has finite logarithmic energy and there exists a constant c such that $2U^\mu(z) + Q(z) = c$ quasi-everywhere on $\operatorname{supp}(\mu)$ *and $2U^\mu(z) + Q(z) \geq c$ quasi-everywhere on $\mathbb{C}$, then $\mu = \sigma_Q$ and $c = c_Q$.*

Remark 2.3.4. (Obstacle function) Let $\check{Q}$ be the maximal subharmonic function $\leq Q$ which grows like $2\log|z| + O(1)$ as z goes to infinity. It is known that

$$\begin{cases} \check{Q}(z) = Q(z) & \text{on} \quad S, \\ \check{Q}(z) = Q^S(z) + 2G(z,\infty) & \text{on} \quad \mathbb{C} \setminus S, \end{cases}$$

where G is the Green's function of $\mathbb{C} \setminus S$ and Q^S is the harmonic extension of $Q|_{\partial S}$ to $\mathbb{C} \setminus S$. We also have $\check{Q} + 2U^\sigma = c_Q$. This $\check{Q}$ is called the *obstacle function*, see [25, 34].

Theorem 2.3.5. *If Q is C^2-smooth, then the equilibrium measure σ is absolutely continuous with respect to the two-dimensional Lebesgue measure and it is of the form*

$$d\sigma = \chi_S \Delta Q\, dA. \tag{2.3.1}$$

Without the smoothness of Q, the formula (2.3.1) is true if the right-hand side is understood in the distributional sense.

2.3.2 Convergence of marginal probability measures

Let $\{\zeta_j\}_{j=1}^n$ be the set of eigenvalues of the random normal matrix ensemble associated with Q, i.e., $\{\zeta_j\}_{j=1}^n$ has the joint distribution $\mathbf{P}_n$ in (2.1.3). To describe the distribution of eigenvalues, we define the k-th *marginal probability measure* $\sigma_{n,k}$ for each $k = 1, 2, \ldots, n$ by

$$\sigma_{n,k}(D) = \mathbf{P}_n[B \times \mathbb{C}^{n-k}],$$

where D is a Borel measurable subset of $\mathbb{C}^k$. We remark that this marginal measure $\sigma_{n,k}$ is associated with the k-point correlation function $\mathbf{R}_{n,k}$ defined in Section 2.2 by the relation

$$d\sigma_{n,k} = \frac{(n-k)!}{n!}\,\mathbf{R}_{n,k}(\zeta_1,\ldots,\zeta_k)\,dV_k(\zeta_1,\ldots,\zeta_k).$$

We also remark that the first marginal measure is the expectation of the empirical measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\zeta_j}$ of eigenvalues, i.e.,

$$\sigma_{n,1}(D) = \mathbf{P}_n[D\times\mathbb{C}^{n-1}] = \frac{1}{n}\mathbb{E}[\#\{j:\zeta_j\in D\}],$$

for Borel measurable sets D in $\mathbb{C}$. Here, $\#$ denotes the counting measure.

Theorem 2.3.6 ([25]). *Let Q be an admissible potential. Suppose that Q is bounded and continuous in a neighborhood of the droplet S_Q. Then we have*

$$\sigma_{n,k}\to\sigma_Q^{\otimes k}$$

as $n\to\infty$ in the weak-star sense of measures, i.e., for continuous and bounded function f defined in $\mathbb{C}^k$,

$$\int_{\mathbb{C}^k} f\,d\sigma_{n,k}\to\int_{\mathbb{C}^k} f\,d\sigma_Q^{\otimes k}\quad\text{as}\quad n\to\infty.$$

It follows by the above theorem that the eigenvalues tend to accumulate in the droplet as n tends to infinity. For one-dimensional case like random Hermitian matrix ensembles, this result was obtained by Johansson in [27].

2.4 Ward's identities

In this section, we present a proof of Ward's identity in [3]. The Ward's identity is the starting point for deriving the rescaled Ward's equation, which plays an important role in proving the universality of scaling limit of eigenvalues. The Ward's identity is also studied in terms of the conformal field theory. For this point of view, see [29], Appendix B.

We first introduce the Ward's identity in [3]. We recall that the joint distribution of eigenvalues of $n\times n$ random normal matrix ensemble associated with Q is given by

$$d\mathbf{P}_n(\boldsymbol{\zeta}) = \frac{1}{Z_n}e^{-H_n(\boldsymbol{\zeta})}dV_n(\boldsymbol{\zeta});\quad Z_n=\int_{\mathbb{C}^n} e^{-H_n(\boldsymbol{\zeta})}\,dV_n(\boldsymbol{\zeta})$$

for $\boldsymbol{\zeta} = (\zeta_j) \in \mathbb{C}^n$, where H_n is the Hamiltonian

$$H_n(\boldsymbol{\zeta}) = -\sum_{j \neq k} \log|\zeta_j - \zeta_k| + n \sum_{j=1}^{n} Q(\zeta_j).$$

Let $\psi \in C_0^\infty(\mathbb{C})$ be a smooth function with compact support. For given random configuration $\{\zeta_j\}_{j=1}^n$ of eigenvalues, we define random variables $I_n[\psi]$, $II_n[\psi]$, $III_n[\psi]$ by

$$I_n[\psi] = \frac{1}{2} \sum_{j \neq k}^{n} \frac{\psi(\zeta_j) - \psi(\zeta_k)}{\zeta_i - \zeta_k} \ ; \quad II_n[\psi] = n \sum_{j=1}^{n} \psi(\zeta_j) \cdot \partial Q(\zeta_j) \ ;$$
$$III_n[\psi] = \sum_{j=1}^{n} \partial \psi(\zeta_j).$$

We write $W_n^+[\psi] = I_n[\psi] - II_n[\psi] + III_n[\psi]$.

Theorem 2.4.1. *Suppose that Q is C^2-smooth in a neighborhood of the support of ψ. Then*

$$\mathbb{E}[W_n^+[\psi]] = 0. \tag{2.4.1}$$

Proof. For a test function $\psi \in C_0^\infty(\mathbb{C})$, let ψ_t denote its flow, i.e., $\psi_t(\zeta_j) = \zeta_j + t\psi(\zeta_j)$ for $|t| < \epsilon$ with $\epsilon > 0$ sufficiently small. Changing the variables $\eta_j = \psi_t(\zeta_j)$, we obtain

$$Z_n = \int_{\mathbb{C}^n} e^{-H_n} \ dV_n = \int_{\mathbb{C}^n} e^{-H_n \circ \Psi_t} \cdot J_t \ dV_n$$

where the flow $\Psi_t : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $(\zeta_j) \mapsto (\eta_j)$ and J_t is the Jacobian of the transformation Ψ_t. For small $|t| \to 0$, we have

$$J_t(\boldsymbol{\zeta}) = \prod_{j=1}^{n} \left(1 + 2t \operatorname{Re} \partial\psi(\zeta_j) + O(|t|^2)\right) = 1 + 2t \operatorname{Re} III_n[\psi] + O(|t|^2).$$

On the other hand, when $|t| \to 0$, we obtain

$$\log|\eta_j - \eta_k|^2 = \log|\zeta_j - \zeta_k|^2 + 2t \operatorname{Re} \frac{\psi(\zeta_j) - \psi(\zeta_k)}{\zeta_j - \zeta_k} + O(|t|^2),$$
$$Q(\eta_j) = Q(\zeta_j) + 2t \operatorname{Re}\left(\psi(\zeta_j) \cdot \partial Q(\zeta_j)\right) + O(|t|^2),$$

which gives

$$(H_n \circ \Psi_t)(\boldsymbol{\zeta}) = H_n(\boldsymbol{\zeta}) + 2t\,(-\operatorname{Re} I_n[\psi] + \operatorname{Re} II_n[\psi]) + O(|t|^2).$$

Hence, we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \int_{\mathbb{C}^n} e^{-H_n \circ \Psi_t} \cdot J_t \, dV_n \\ &= \int_{\mathbb{C}^n} 2 \operatorname{Re}(I_n[\psi] - II_n[\psi] + III_n[\psi]) e^{-H_n(\zeta)} \, dV_n(\boldsymbol{\zeta}), \end{aligned}$$

which implies that $\mathbb{E}[\operatorname{Re} W_n^+[\psi]] = 0$. Repeating the above argument for $i\psi$ instead of ψ, we obtain $\mathbb{E}[\operatorname{Im} W_n^+[\psi]] = 0$ and hence, $\mathbb{E}[W_n^+[\psi]] = 0$. $\square$

Remark 2.4.2. (β-ensembles) For the β-ensemble defined in Remark 2.1.2, the following Ward's identity holds: for $\psi \in C_0^\infty(\mathbb{C})$,

$$\mathbb{E}[\, \beta(I_n[\psi] - II_n[\psi]) + III_n[\psi]\,] = 0.$$

Chapter 3

Rescaled point processes near a bulk singularity

In this chapter, we study the scaling limits for the eigenvalues at a bulk singularity. Precisely, we obtain a proper microscopic scale for each bulk singularity and prove the existence and universality of the limit of the rescaled point process near the bulk singularity. This chapter is based on [6].

3.1 Introduction

Consider the eigenvalues $\{\zeta_j\}_{j=1}^n$ of the random normal ensemble associated with the potential Q. We assume in this chapter that Q is admissible and real-analytic in the interior of the set $\{Q < \infty\}$.

We recall that the marginal probability measure of the eigenvalues converges to the equilibrium measure σ_Q associated with Q. We focus on the case when the measure σ_Q vanishes at an isolated point p in the interior of droplet S_Q, i.e., $\Delta Q(p) = 0$.

Definition 3.1.1. Let $p \in \operatorname{Int} S_Q$ be a point where $\Delta Q(p) = 0$ and d be an integer with $d > 1$. We say the potential Q has a *bulk singularity of the type* $2d-2$ at p if the Taylor expansion of ΔQ about p is of the form

$$\Delta Q(\zeta) = P(\zeta - p) + O(|\zeta - p|^{2d-1}),$$

where $P(x + iy)$ is a homogeneous polynomial in x, y of degree $2d-2$ and P is positive definite, i.e., $P(\zeta) > 0$ for every $\zeta \neq 0$.

For now, we assume that $0 \in \operatorname{Int} S_Q$ and Q has a bulk singularity at 0 of type $2d-2$.

3.1.1 Microscopic scale

We first define a proper scale to investigate the microscopic properties of the eigenvalues $\{\zeta_j\}$ at the bulk singularity $p = 0$. The *microscopic scale* $r_n = r_n(p)$ at the point p is defined by the equation

$$n \int_{D(p,r_n)} \Delta Q \, dA = 1. \tag{3.1.1}$$

This r_n is the expected distance between p and other eigenvalues. Indeed, as in Section 2.3.2, the expected empirical measure $\sigma_{n,1}$,

$$\sigma_{n,1}(D) = \frac{1}{n}\mathbb{E}[\#\{j : \zeta_j \in D\}] \quad \text{for all Borel} \quad D \subset \mathbb{C},$$

converges to the equilibrium measure $d\sigma = \chi_S \Delta Q \, dA$. This means that the expected number of eigenvalues which fall in $D(p, r_n)$ is asymptotically equal to one.

Proposition 3.1.2. *Let Q have a bulk singularity of the type $2d-2$ at the origin. Then the microscopic scale $r_n = r_n(0)$ satisfies*

$$r_n = \tau_0 \, n^{-1/2d}(1 + O(n^{-1/2d})),$$

as $n \to \infty$, where τ_0 is the constant satisfying

$$\tau_0^{-2d} = \frac{1}{2\pi d}\int_0^{2\pi} P(e^{i\theta})d\theta. \tag{3.1.2}$$

We call τ_0 the *modulus* of the bulk singularity at 0.

Proof. We have for all $z \in D(0, r_n)$,

$$\Delta Q(\zeta) = (1 + O(r_n)) \; P(\zeta).$$

Since P is a homogeneous polynomial, the equation (3.1.1) implies that

$$\begin{aligned} 1 &= n(1+O(r_n)) \int_{D(0,\,r_n)} P(\zeta)\, dA(\zeta) \\ &= n(1+O(r_n))\, \frac{1}{\pi} \int_0^{r_n} \int_0^{2\pi} r^{2d-2} P(e^{i\theta})\, r\, d\theta\, dr. \end{aligned}$$

Thus, we have $r_n^{2d} = \tau_0^{2d}\, n^{-1}\,(1+O(r_n))$ and it proves the proposition. $\square$

3.1.2 Rescaled point processes

Let $\{\zeta_j\}_{j=1}^n$ be the system of eigenvalues of the random normal matrix ensemble associated with Q. Recall that the joint distribution of the system is given by (2.1.3) and it forms a determinantal point process. See Section 2.2. We denote the k-point correlation function and the correlation kernel of the process by $\mathbf{R}_{n,k}$ and $\mathbf{K}_n$, respectively. We also denote the one-point function by $\mathbf{R}_n = \mathbf{R}_{n,1}$.

We now consider the rescaled point process $\Theta_n = \{z_j\}_1^n$ defined by

$$z_j = r_n^{-1}\zeta_j \quad \text{for} \quad j = 1, \ldots, n.$$

That is, the point process Θ_n is obtained by rescaling the eigenvalues ζ_j about the bulk singularity 0 on the microscopic scale r_n. Obviously, Θ_n forms a determinantal point process and its k-point correlation function is given by

$$R_{n,k}(z_1, \ldots, z_k) = r_n^{2k}\, \mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k).$$

The correlation kernel K_n of the rescaled point process Θ_n is given by

$$K_n(z,w) = r_n^2\, \mathbf{K}_n(\zeta, \eta),$$

where $z = r_n^{-1}\zeta$ and $w = r_n^{-1}\eta$.

3.1.3 Canonical decomposition

We are interested in the limit of the rescaled correlation kernel. We find out a dominant part of the potential Q which determines the structure of the limiting kernel. Let $P_{2d}(\zeta, \bar{\zeta})$ be the Taylor polynomial of Q of degree $2d$ about the origin. To extract the harmonic part, we define a holomorphic polynomial H by

$$H(\zeta) = Q(0) + 2\partial Q(0)\cdot\zeta + \partial^2 Q(0)\cdot\zeta^2 + \cdots + \frac{2}{(2d)!}\partial^{2d}Q(0)\cdot\zeta^{2d}.$$

Now we denote by Q_0 the polynomial $P_{2d} - \operatorname{Re} H$. Then Q_0 is the homogeneous polynomial of degree $2d$ by the definition 3.1.1. We refer to Q_0 the dominant part of Q at the origin. The canonical decomposition of Q is given by

$$Q = Q_0 + \operatorname{Re} H + Q_1,$$

where $Q_1(\zeta) = O(|\zeta|^{2d+1})$ as $\zeta \to 0$.

Example 3.1.3. Consider the potential $Q(\zeta) = |\zeta|^{2d} - c\operatorname{Re}\zeta^{2d}$ where $|c| < 1$. From the above decomposition, the dominant part of Q is $|\zeta|^{2d}$.

3.1.4 Example : The Mittag-Leffler ensembles

Before introducing the main theorems, we first give a motivated example and some intuitive results. Consider the random normal matrix ensemble associated with the power potential $Q(\zeta) = |\zeta|^{2d}$ where $d \geq 1$ is an integer. We note that Q has a bulk singularity of the type $2d-2$ at the origin.

Orthonormal polynomials with respect to the measure $e^{-nQ}dA$ can be calculated explicitly. Since the measure is radially symmetric, we can take orthogonal polynomials to be monomials. Thus, the correlation kernel of the process of eigenvalues is obtained as follows:

$$\mathbf{K}_n(\zeta,\eta) = dn^{\frac{1}{d}} \sum_{j=0}^{n-1} \frac{\left(n^{\frac{1}{d}}\zeta\bar{\eta}\right)^j}{\Gamma\left(\frac{j+1}{d}\right)} e^{-n\left(|\zeta|^{2d}+|\eta|^{2d}\right)/2},$$

where Γ is the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt.$$

By (3.1.1), the microscopic scale r_n at 0 is $r_n = (dn)^{-\frac{1}{2d}}$. If we rescale via $z = r_n^{-1}\zeta$ and $w = r_n^{-1}\eta$, then the rescaled kernel is given by

$$K_n(z,w) = d^{-\frac{1}{d}+1} \sum_{j=1}^{n-1} \frac{\left(d^{-\frac{1}{d}}z\bar{w}\right)^j}{\Gamma\left(\frac{j+1}{d}\right)} e^{-\frac{1}{d}\left(|z|^{2d}+|w|^{2d}\right)/2}.$$

Let $E_{a,b}$ be the Mittag-Leffler function defined by

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}. \tag{3.1.3}$$

Obviously, K_n converges to

$$K(z,w) = d^{-\frac{1}{d}+1} E_{\frac{1}{d},\frac{1}{d}} \left(d^{-\frac{1}{d}} z\bar{w} \right) e^{-\frac{1}{d}\left(|z|^{2d}+|w|^{2d}\right)/2} \tag{3.1.4}$$

locally uniformly in $\mathbb{C}^2$ as $n \to \infty$. It also implies the locally uniform convergence of k-point correlation functions for each $k = 1, 2, \cdots$. Hence, we conclude that the rescaled point process $\{z_j\}_1^n$ converges to the point process with correlation kernel $K(z,w)$.

3.2 Main results

Let Q be a potential which has a bulk singularity at the origin of type $2d-2$, and write its canonical decomposition as $Q = Q_0 + \operatorname{Re} H + Q_1$. We denote the rescaled process by Θ_n with the microscopic scale r_n and its correlation kernel by K_n. We are interested in the limit of Θ_n.

Structure of limiting Kernels

The following theorem shows the existence of the limiting kernel and its structure. Before stating the theorem, we require some definitions. A continuous function $h(z,w)$ defined in $\mathbb{C}^2$ is called *Hermitian* if $h(z,w) = \overline{h(w,z)}$. A hermitian function h is called *Hermitian-entire* if $h(z,w)$ is entire in z and $\bar{w}$. A Hermitian function $c(z,w)$ is called a *cocycle* if there is a unimodular function g such that $c(z,w) = g(z)\overline{g(w)}$. Note that the cocycle function has a property

$$c(z_1,z_2)\, c(z_2,z_3) \cdots c(z_{k-1},z_k)\, c(z_k,z_1) = 1$$

for all $k \geq 1$ and all $z_1, \cdots, z_k \in \mathbb{C}$.

Theorem 3.2.1. *There exists a sequence $\{c_n\}$ of cocycles such that every subsequence of $c_n K_n$ has a subsequence which is locally convergent in $\mathbb{C}^2$. Every limit point K of $c_n K_n$ is Hermitian and*

$$K(z,w) = L(z,w)\, e^{-(Q_0(\tau_0 z)+Q_0(\tau_0 w))/2},$$

where L is Hermitian-entire and τ_0 is the modulus of Q defined by (3.1.2)*.*

We call the function L a *limiting holomorphic kernel*, and we write a limiting one point function $R(z) = K(z, z)$.

We remark that we cannot exclude the possibility that R is identically zero at this moment. In fact, the trivial case occurs when the point process is rescaled at the point in the exterior of droplet. However, an estimate of R given in Chapter 5 ensures the non-triviality of R when the system is rescaled at a singularity in the bulk.

The Ward's equation

Before we introduce the Ward's equation, we define the following functions:

$$B(z, w) := \frac{|K(z, w)|^2}{K(z, z)} \quad \text{and} \quad C(z) := \int \frac{B(z, w)}{z - w} dA(w)$$

defined at the point where $K(z, z) > 0$. We call $B(z, w)$ a *limiting Berezin kernel* rooted at z and $C(z)$ the Cauchy transform of a kernel B. The following theorem, so called "zero-one law", asserts the positivity of the one point function.

Theorem 3.2.2. *Let R be a limiting one point function. Then R is either positive everywhere or identically zero.*

We have the following *Ward's equation*:

Theorem 3.2.3. *If R is non-trivial, then for $z \in \mathbb{C}$,*

$$\bar{\partial} C(z) = R(z) - \tau_0^2 \Delta Q_0(\tau_0 z) - \Delta \log R(z).$$

Reproducing kernels in Bergman spaces

We consider the measure $d\mu_0(z) = e^{-Q_0(\tau_0 z)} dA(z)$ on $\mathbb{C}$ and the Bergman space $L_a^2(\mu_0)$ consisting of all holomorphic functions u defined on $\mathbb{C}$ such that the following L^2-norm is finite:

$$\| u \|_{L^2(\mu_0)}^2 = \int_{\mathbb{C}} |u|^2 \, d\mu_0 < \infty.$$

We denote by L_0 the Bergman kernel of the space $L^2_a(\mu_0)$. Then we have

$$L_0(z,w) = \sum_{j=0}^{\infty} e_j(z)\overline{e_j(w)}$$

where e_j is an orthonormal polynomial of degree j with respect to the measure μ_0.

Example 3.2.4. If $Q_0(z) = |z|^{2d}$ $(d \geq 1)$, then the modulus τ_0 is $d^{-\frac{1}{2d}}$ and orthonormal polynomials are $e_j(z) = z^j / \| z^j \|_{L^2(\mu_0)}$ where

$$\| z^j \|^2_{L^2(\mu_0)} = \int_{\mathbb{C}} |z|^{2j} \, e^{-|\tau_0 z|^{2d}} \, dA = d^{\frac{j+1}{d}-1} \Gamma\left(\frac{j+1}{d}\right).$$

Hence, the Bergman kernel L_0 is given by

$$L_0(z,w) = d^{-\frac{1}{d}+1} E_{\frac{1}{d},\frac{1}{d}} \left(d^{-\frac{1}{d}} z\bar{w}\right)$$

where $E_{a,b}$ is the Mittag-Leffler function (3.1.3). Comparing L_0 with $K(z,w) = L(z,w)e^{-Q_0(\tau z)/2 - Q_0(\tau w)/2}$ in (3.1.4) for the Mittag-Leffler ensemble, we obtain $L = L_0$.

Universality for dominant radial singularities

With the canonical decomposition $Q = Q_0 + \operatorname{Re} H + Q_1$, we say a bulk singularity is *dominant radial* if Q_0 is radially symmetric, i.e. $Q_0(z) = Q_0(|z|)$. A limiting holomorphic kernel L is said to be *rotationally symmetric* if $L(z,w) = L(ze^{it}, we^{it})$ for all real t.

Theorem 3.2.5. *Suppose that Q has a bulk singularity at* 0 *which is dominant radial. If a limiting holomorphic kernel L is rotationally symmetric, then $L = L_0$.*

Universality for homogeneous singularities

Let $Q = Q_0 + \operatorname{Re} H + Q_1$ be the canonical decomposition. We say a bulk singularity at 0 of type $2d-2$ is *homogeneous* if $Q_1 = 0$ and $H(\zeta) = \alpha\zeta^{2d}$ for some α, i.e., Q is homogeneous of degree $2d$.

Theorem 3.2.6. *Suppose that Q has a bulk singularity at* 0 *which is homogeneous. Then every limiting holomorphic kernel L is equal to L_0.*

Convention

From now on, we assume that the microscopic scale r_n satisfies $n\, r_n^{2d} = 1$, i.e., $\tau_0 = 1$ for simplicity. It can be achieved by multiplying Q by some constant.

3.3 Existence of limiting kernels

We first introduce some spaces of weighted polynomials. We write $d\mu_n = e^{-nQ}\, dA$ and denote by $L^2(\mu_n)$ the space of functions on $\mathbb{C}$ equipped with the norm

$$\| f \|^2_{L^2(\mu_n)} = \int_{\mathbb{C}} |f|^2\, e^{-nQ}\, dA.$$

Let $\mathcal{P}_n$ be the subspace of $L^2(\mu_n)$ consisting of all holomorphic polynomials of degree less than n. We define $\mathcal{W}_n$ to be the space of all weighted polynomials of degree less than n,

$$\mathcal{W}_n = \{p \cdot e^{-nQ} \,:\, p \in \mathcal{P}_n\}.$$

Noting that each element $p \cdot e^{-nQ} \in \mathcal{W}_n$ has a finite (usual) L^2-norm, we consider $\mathcal{W}_n$ as a subspace of $L^2 = L^2(dA)$. Recall that the correlation kernel $\mathbf{K}_n$ of the processes is of the form

$$\mathbf{K}_n(\zeta, \eta) = \mathbf{k}_n(\zeta, \eta)\, e^{-n(Q(\zeta)+Q(\eta))/2},$$

where $\mathbf{k}_n$ is the reproducing kernel of the space $\mathcal{P}_n$. The correlation kernel $\mathbf{K}_n$ can be regarded as the reproducing kernel of the space $\mathcal{W}_n$.

Now consider the rescaled point process. We write the rescaled potential $\tilde{Q}_n(z) = nQ(r_n z)$ and the corresponding measure $d\tilde{\mu}_n = e^{-\tilde{Q}_n}\, dA$. We denote by $\tilde{\mathcal{P}}_n$ and $\tilde{\mathcal{W}}_n$ the space of holomorphic polynomials and the space of weighted polynomials with respect to $\tilde{\mu}_n$, respectively. Then the correlation kernels

$$k_n(z, w) = r_n^2 \mathbf{k}_n(\zeta, \eta) \quad \text{and} \quad K_n(z, w) = r_n^2 \mathbf{K}_n(\zeta, \eta)$$

with $z = r_n^{-1}\zeta$ and $w = r_n^{-1}\eta$ are reproducing kernels for the space $\tilde{\mathcal{P}}_n$ and $\tilde{\mathcal{W}}_n$, respectively.

3.3.1 Local uniform boundedness of the rescaled kernel

Lemma 3.3.1. *For each compact subset V of $\mathbb{C}$, there exists a constant $C = C(V)$ such that for all $u \in L^2(\tilde{\mu}_n)$ and $z \in V$,*

$$|u(z)|^2 \, e^{-\tilde{Q}_n(z)} \le C \, \| \, u \, \|_{L^2(\tilde{\mu}_n)}^2 \, .$$

Here, the constant C is uniform on n.

Proof. Fix a positive number δ. For a compact set V of $\mathbb{C}$, we write

$$V_\delta = \{z \in \mathbb{C} \, : \, \mathrm{dist}(z, V) \le \delta\}.$$

We choose a number α satisfying $\alpha > \sup\{\Delta Q_0(z) \, : \, z \in V_\delta\}$. Then we have $\alpha > \Delta \tilde{Q}_n(z)$ for sufficiently large n and all $z \in V_\delta$ since

$$\Delta \tilde{Q}_n(z) = n r_n^2 (\Delta Q_0(r_n z) + \Delta Q_1(r_n z)) = n r_n^{2d} \Delta Q_0(z) + O(n r_n^{2d+1})$$

where $n r_n^{2d} = 1$. Consider a function

$$F_n(z) = u(z) \, e^{-\tilde{Q}_n(z)/2 + \alpha |z|^2/2}.$$

Then $\Delta \log |F_n|^2 \ge -\Delta \tilde{Q}_n(z) + \alpha > 0$ for all $z \in V_\delta$ and sufficiently large n, whence $|F_n|^2$ is subharmonic in V_δ. By the sub-mean value property, we have for $z \in V$,

$$|F_n(z)|^2 \le \delta^{-2} \int_{D(z,\delta)} |F_n(w)|^2 \, dA(w).$$

Thus we obtain

$$\begin{aligned} |u(z)|^2 \, e^{-\tilde{Q}_n(z)} &\le \delta^{-2} e^{\alpha(|z|+\delta)^2 - \alpha |z|^2} \int_{D(z,\delta)} |u|^2 \, e^{-\tilde{Q}_n} \, dA \\ &\le \delta^{-2} e^{\alpha(2 M_V \delta + \delta^2)} \, \| \, u \, \|_{L^2(\tilde{\mu}_n)}^2 \end{aligned}$$

where $M_V = \sup\{|z| \, : \, z \in V\}$, which proves the lemma. ∎

Lemma 3.3.2. *The family of K_n is locally uniformly bounded in $\mathbb{C}^2$.*

Proof. The rescaled kernel k_n is the reproducing kernel for the space $\tilde{\mathcal{P}}_n$.

Thus we can write

$$k_n(z,z) = \sup\{|p(z)|^2 \,:\, p \in \tilde{\mathcal{P}}_n,\ \|\, p\,\|_{L^2(\tilde{\mu}_n)} \le 1\}.$$

Since $K_n(z,z) = k_n(z,z)e^{-\tilde{Q}_n}$, the family of $K_n(z,z)$ is locally uniformly bounded by Lemma 3.3.1. The Cauchy-Schwarz inequality,

$$|K_n(z,w)|^2 \le K_n(z,z)K_n(w,w),$$

gives the local uniform boundedness of K_n on $\mathbb{C}^2$. □

3.3.2 Structure of limiting kernels

Recall the canonical decomposition $Q = Q_0 + \operatorname{Re} H + Q_1$. Then

$$nQ(r_n z) = Q_0(z) + \operatorname{Re} H_n(z) + Q_{1,n}(z)$$

where $H_n(z) = nH(r_n z)$ and $Q_{1,n}(z) = nQ_1(r_n z)$. We define a Hermitian-entire function by

$$L_n(z,w) = k_n(z,w)\, e^{-(H_n(z)+\bar{H}_n(w))/2}.$$

We refer to L_n the *rescaled holomorphic kernel.* The following lemma implies Theorem 3.2.1.

Lemma 3.3.3. *Each subsequence of L_n has a locally uniformly convergent subsequence and each limit L is Hermitian-entire. Moreover, there exists a sequence of cocycles c_n such that every subsequence of $c_n K_n$ has a subsequence converging to a Hermitian function K locally uniformly on $\mathbb{C}^2$ and each limit K is of the form $K(z,w) = L(z,w)\, e^{-(Q_0(z)+Q_0(w))/2}$.*

Proof. We write $K_n = L_n E_n$ where

$$E_n(z,w) = e^{(H_n(z)+\bar{H}_n(w)-nQ(r_n z)-nQ(r_n w))/2}.$$

Recall that the holomorphic function H is defined by

$$H(\zeta) = Q(0) + 2\partial Q(0)\cdot\zeta + \partial^2 Q(0)\cdot\zeta^2 + \cdots + \frac{2}{(2d)!}\partial^{2d}Q(0)\cdot\zeta^{2d},$$

so that

$$(H_n(z) - H_n(w))/2 = n \sum_{j=1}^{2d} r_n^j \frac{\partial^j Q(0)}{j!} \cdot (z^j - w^j). \tag{3.3.1}$$

Now we define

$$c_n(z,w) := e^{i \operatorname{Im}(H_n(z) - H_n(w))/2}$$

and by (3.3.1), $c_n(z,w) = g(z)\bar{g}(w)$ for some unimodular function g, i.e, $c_n(z,w)$ is a cocycle. Noting that

$$\begin{aligned} &H_n(z) + \bar{H}_n(w) - nQ(r_n z) - nQ(r_n w) \\ &= -Q_0(z) - Q_0(w) - Q_{n,1}(z) - Q_{n,1}(w) + i(\operatorname{Im} H_n(z) - \operatorname{Im} H_n(w)) \end{aligned}$$

and $Q_{n,1}(z) = O(n\,(r_n)^{2d+1})$ uniformly on each compact subset as $n \to \infty$, we obtain that

$$c_n(z,w) E_n(z,w) = e^{-(Q_0(z)+Q_0(w))/2}(1 + o(1)) \tag{3.3.2}$$

where $o(1) \to 0$ uniformly on each compact subset of $\mathbb{C}^2$ as $n \to \infty$. From this, we also see that the functions $|E_n(z)|^{-1}$ have a uniform bound for each compact set of $\mathbb{C}^2$. Hence, by Lemma 3.3.2, the family of $L_n = K_n E_n^{-1}$ is locally uniformly bounded, so that $\{L_n\}$ is a normal family of Hermitian-entire functions. Thus we can choose a subsequence $\{L_{n_l}\}$ which converges to a Hermitian-entire function L uniformly on each compact subset of $\mathbb{C}^2$. Finally, we have the convergence

$$c_{n_l}(z,w)\, K_{n_l}(z,w) \to L(z,w)\, e^{-(Q_0(z)+Q_0(w))/2}$$

which is uniform on each compact subset of $\mathbb{C}^2$. The lemma is proved. $\square$

We call a limit L a *limiting holomorphic kernel*. The next result is so-called the"mass-one inequality":

Lemma 3.3.4. *Suppose that $L_{n_l} \to L$ as in Lemma 3.3.3. Then a limit L satisfies that for $z \in \mathbb{C}$,*

$$\int_{\mathbb{C}} |L(z,w)|^2\, e^{-Q_0(w)}\, dA(w) \le L(z,z). \tag{3.3.3}$$

Proof. Since the correlation kernel K_n is the reproducing kernel of the space $\tilde{\mathcal{W}}_n$, K_n satisfies the reproducing property

$$\int |K_{n_l}(z,w)|^2 \, dA(w) = K_{n_l}(z,z).$$

It can be written again as

$$\int \left| L_{n_1}(z,w) \, E_{n_l}(z,w) \right|^2 \, dA(w) = L_{n_l}(z,z) \, E_{n_l}(z,z).$$

Letting $l \to \infty$, we prove the lemma by (3.3.2) and Fatou's lemma. $\square$

3.4 Properties of limiting holomorphic kernels

In this section, we prove some Lemmas for a limiting holomorphic kernel.

3.4.1 Positive matrices and reproducing kernels

First, we give some results about the reproducing kernel Hilbert spaces.

Definition 3.4.1. A Hermitian function $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is called a **positive matrix** if

$$\sum_{i,j=1}^{N} \alpha_i \bar{\alpha}_j F(z_i, z_j) \geq 0$$

for all points $z_j \in \mathbb{C}$ and all scalars $\alpha_j \in \mathbb{C}$.

The following theorem is well-known in the theory of reproducing kernels. See [8].

Theorem 3.4.2 (Moore). *If a Hermitian function $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a positive matrix, then there exists a reproducing kernel Hilbert space $\mathcal{H}_F$ of functions on $\mathbb{C}$ such that F is the reproducing kernel of $\mathcal{H}_F$.*

More precisely, we construct the Hilbert space $\mathcal{H}_F$ in the following way. Let $F_z(w) = F(w,z)$ for each $z \in \mathbb{C}$ and let $\mathcal{M}$ be the space spanned by the set $\{F_z : z \in \mathbb{C}\}$. Then we define a positive semi-definite inner product on $\mathcal{M}$ by $\left\langle \sum_i \alpha_i F_{z_i}, \sum_j \beta_j F_{z_j} \right\rangle = \sum_{i,j} \alpha_i \bar{\beta}_j F(z_j, z_i)$, where α_i and β_j are scalars in $\mathbb{C}$. If we denote by $\mathcal{H}_F$ the completion of $\mathcal{M}$ with respect to the inner product, then we obtain a Hilbert space $\mathcal{H}_F$ whose reproducing kernel is F.

The converse is rather elementary, so we give a short proof.

Proposition 3.4.3. *Let $\mathcal{H}$ be a Hilbert space with reproducing kernel F. Then F is a positive matrix.*

Proof. Fix some points $z_1, \dots, z_m \in \mathbb{C}$ and some scalars $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. We obtain that

$$\sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j F(z_i, z_j) = \left\langle \sum_i \alpha_i F_{z_i}\,,\, \sum_j \alpha_j F_{z_j} \right\rangle = \left\| \sum_k \alpha_k F_{z_k} \right\|^2 \geq 0,$$

which proves the proposition. □

Now, we are ready to prove our lemmas. Let L be a limiting holomorphic kernel. We write $L_z(w) := L(w, z)$.

Lemma 3.4.4. *Let $\mathcal{H}_L$ be the completion of the linear span of $\{L_z\}_{z\in\mathbb{C}}$, equipped with the inner product induced by the assignment $\langle L_z,\, L_w\rangle = L(w, z)$. Then L is a reproducing kernel for $\mathcal{H}_L$.*

Proof. We note that K_n is the reproducing kernel of the space $\tilde{\mathcal{W}}_n$, a subspace of $L^2(dA)$. By Proposition 3.4.3, K_n and every limit point K are positive matrices. Since $L(z, w) = K(z, w)\, e^{Q_0(z)/2+Q_0(w)/2}$, we obtain

$$\sum_{j,k} \alpha_j \bar{\alpha}_k L(z_j, z_k) = \sum_{j,k} \alpha_j \bar{\alpha}_k K(z_j, z_k) e^{Q_0(z_j)/2}\, e^{Q_0(z_k)/2} \geq 0,$$

for each point $z_j \in \mathbb{C}$ and each scalar $\alpha_j \in \mathbb{C}$. The second statement is a direct consequence of Theorem 3.4.2. □

Lemma 3.4.5. *The function $z \mapsto L(z, z)$ is logarithmic subharmonic.*

Proof. By Lemma 3.4.4, L is a reproducing kernel of the Hilbert space $\mathcal{H}_L$ defined in the lemma. Then, $\bar{\partial}_z L_z \in \mathcal{H}_L$ since L is Hermitian-entire. We have $\bar{\partial}_z L(z, z) = \langle \bar{\partial}_z L_z,\, L_z\rangle$ and $\Delta_z L(z, z) = \langle \bar{\partial}_z L_z,\, \bar{\partial}_z L_z\rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product on $\mathcal{H}_L$ defined in Lemma 3.4.4. Thus we obtain that at the point where $L(z, z) > 0$,

$$\begin{aligned}\Delta_z \log L(z, z) &= \frac{L(z, z)\Delta_z L(z, z) - \partial_z L(z, z)\, \bar{\partial}_z L(z, z)}{L(z, z)^2} \\ &= \frac{\langle L_z,\, L_z\rangle \cdot \langle \bar{\partial}_z L_z,\, \bar{\partial}_z L_z\rangle - \left|\langle \bar{\partial}_z L_z,\, L_z\rangle\right|^2}{L(z, z)^2},\end{aligned}$$

which is non-negative by the Cauchy-Schwarz inequality.

If $L(z,z)=0$, then $\log L(z,z)=-\infty$. Thus, $\log L(z,z)$ satisfies the sub-mean value property at the point when $L(z,z)=0$. Hence, the proof is complete. $\square$

3.4.2 The positivity theorem

In this section, we prove the "positivity" theorem. As we see in Lemma 3.4.4, a limiting holomorphic kernel L is a reproducing kernel for the Hilbert space $\mathcal{H}_L$ which is the completion of the linear span $\mathcal{M}$ of L_z's. For the measure $d\mu_0 = e^{-Q_0}\,dA$, the mass-one inequality (3.3.4) gives

$$\| L_z \|^2_{L^2(\mu_0)} = \int |L(w,z)|^2\, e^{-Q_0(w)}\, dA(w) \le L(z,z) = \| L_z \|^2_{\mathcal{H}_L},$$

where $\langle L_z, L_w\rangle_{\mathcal{H}_L} = L(w,z)$. Thus, $\mathcal{M}$ is contained in $L^2(\mu_0)$ and the inclusion map from $\mathcal{M}$ to $L^2(\mu_0)$ is a contraction. Obviously, the completion $\mathcal{H}_L$ of $\mathcal{M}$ is contractively embedded in $L^2_a(\mu_0)$.

By Aronszajn [8], $L_0 - L$ is a positive matrix where L_0 is the reproducing kernel for $L^2_a(\mu_0)$. Hence, the following theorem holds.

Theorem 3.4.6. *A limiting holomorphic kernel L is a reproducing kernel of a contractively embedded subspace of $L^2_a(\mu_0)$ and $L_0 - L$ is a positive matrix.*

3.5 Ward's equation and zero-one law

In this section, we prove the Theorem 3.2.2 and Theorem 3.2.3. We assume that $\tau_0 = 1$ for simplicity. To begin with, we recall the definitions and give some new definitions.

Let K and L be a limiting correlation kernel and a limiting holomorphic kernel in Lemma 3.3.3. Then, a limiting Berezin kernel rooted at z and its Cauchy transform are defined by

$$B(z,w) = \frac{|K(z,w)|^2}{K(z,z)}, \quad C(z) = \int \frac{B(z,w)}{z-w}\, dA(w) \tag{3.5.1}$$

at a point where $K(z,z) > 0$. We write $R(z) = K(z,z) = L(z,z)e^{-Q_0(z)}$. Now we consider a rescaled system of eigenvalues $\{z_j\}_1^n$ with rescaling via $z_j = r_n^{-1}\zeta_j$. We define a Berezin kernel rooted at z and its Cauchy transform by

$$B_n(z,w) = \frac{|K_n(z,w)|^2}{K_n(z,z)}, \quad C_n(z) = \int \frac{B_n(z,w)}{z-w}\, dA(w). \tag{3.5.2}$$

R_n denotes the rescaled one-point function $R_{n,1}$.

We restate Theorem 3.2.2 and Theorem 3.2.3 in terms of a limiting holomorphic function $L(z,z)$ for the case when $\tau_0 = 1$. We prove the following Lemma in this section.

Lemma 3.5.1. *If $L(z,z)$ does not vanish identically, then $L(z,z)$ is positive everywhere and for $z \in \mathbb{C}$,*

$$\bar{\partial} C(z) = L(z,z)\, e^{-Q_0(z)} - \Delta_z \log L(z,z). \tag{3.5.3}$$

3.5.1 The rescaled Ward's equation

We recall the following Ward's inequality proved in Section 2.4. For a test function $\psi \in C_0^\infty(\mathbb{C})$ and a system of eigenvalues $\{\zeta_j\}_{j=1}^n$,

$$\mathbb{E}[I_n[\psi] - II[\psi] + III[\psi]] = 0, \tag{3.5.4}$$

where

$$I_n[\psi] = \frac{1}{2}\sum_{j\neq k}^n \frac{\psi(\zeta_j)-\psi(\zeta_k)}{\zeta_i - \zeta_k}\ ; \quad II_n[\psi] = n\sum_{j=1}^n \psi(\zeta_j)\cdot \partial Q(\zeta_j)\ ;$$
$$III_n[\psi] = \sum_{j=1}^n \partial\psi(\zeta_j).$$

The following version of Ward's equation is derived from the Ward's identity (3.5.4).

Lemma 3.5.2.

$$\bar{\partial} C_n(z) = R_n(z) - \Delta Q_0(z) - \Delta \log R_n(z) + o(1), \tag{3.5.5}$$

where $o(1) \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$.

Proof. Fix a test function ψ and write $\psi_n(\zeta) = \psi(r_n^{-1}\zeta)$. The Ward's identity (3.5.4) holds for ψ_n, i.e, $\mathbb{E}[I_n[\psi_n] - II[\psi_n] + III[\psi_n]] = 0$. We now calculate the expectations. We recall that $\mathbf{P}_n$ denotes the joint distribution (2.1.3) of eigenvalues and write

$$d\mathbf{P}_n(\zeta_1, \cdots, \zeta_n) = \boldsymbol{p}_n(\zeta_1, \cdots, \zeta_n)\, dA^{\otimes n}(\zeta_1, \cdots, \zeta_n).$$

We also note that the k-point correlation function $\mathbf{R}_{n,k}$ satisfies that

$$\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k) = \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} \boldsymbol{p}_n(\zeta_1, \ldots, \zeta_n)\, dA(\zeta_{k+1}) \cdots dA(\zeta_n).$$

Then we first obtain

$$\begin{aligned} \mathbb{E}[I_n[\psi_n]] &= \int_{\mathbb{C}^n} I_n[\psi_n]\, d\mathbf{P}_n \\ &= \frac{1}{2} \iint_{\mathbb{C}^2} \frac{\psi_n(\zeta) - \psi_n(\eta)}{\zeta - \eta}\, \mathbf{R}_{n,2}(\zeta, \eta)\, dA(\zeta) dA(\eta). \end{aligned}$$

By changing variables $z = r_n^{-1}\zeta$, $w = r_n^{-1}\eta$,

$$\mathbb{E}[I_n[\psi_n]] = r_n^{-1} \int_{\mathbb{C}} \psi(z)\, dA(z) \int_{\mathbb{C}} \frac{R_{n,2}(z,w)}{z-w}\, dA(w).$$

The second term is calculated in the same way,

$$\begin{aligned} \mathbb{E}[II_n[\psi_n]] &= n \int_{\mathbb{C}} \psi_n(\zeta)\, \partial Q(\zeta)\, \mathbf{R}_{n,1}(\zeta)\, dA(\zeta) \\ &= n \int_{\mathbb{C}} \psi(z)\, \partial Q(r_n z)\, R_{n,1}(z)\, dA(z). \end{aligned}$$

The third is obtained as follows:

$$\begin{aligned} \mathbb{E}[III_n[\psi_n]] &= \int_{\mathbb{C}} \partial\psi_n(\zeta)\, \mathbf{R}_{n,1}(\zeta)\, dA(\zeta) = r_n^{-1} \int_{\mathbb{C}} \partial\psi(z)\, R_{n,1}(z)\, dA(z) \\ &= -r_n^{-1} \int_{\mathbb{C}} \psi(z)\, \partial R_{n,1}(z)\, dA(z). \end{aligned}$$

The last equality is obtained by integration by parts. Thus, the Ward's identity gives

$$\int_{\mathbb{C}} \psi(z) \left[\int_{\mathbb{C}} \frac{R_{n,2}(z,w)}{z-w}\, dA(w) - n r_n \partial Q(r_n z) R_{n,1}(z) - \partial R_{n,1}(z) \right] dA(z) = 0.$$

Hence, we conclude that the equation

$$\int_{\mathbb{C}} \frac{R_{n,2}(z,w)}{z-w}\, dA(w) = n r_n \partial Q(r_n z) R_{n,1}(z) + \partial R_{n,1}(z) \tag{3.5.6}$$

holds in the sense of distribution. Note that

$$R_{n,2}(z,w) = R_{n,1}(z)R_{n,1}(w) - |K_n(z,w)|^2 .$$

Dividing each side of (3.5.6) by $R_{n,1}(z)$, we obtain

$$\int_{\mathbb{C}} \frac{R_{n,1}(w)}{z-w} dA(w) - \int_{\mathbb{C}} \frac{B_n(z,w)}{z-w} dA(w) = nr_n \partial Q(r_n z) + \frac{\partial R_{n,1}(z)}{R_{n,1}(z)}.$$

Differentiating each side with $\bar{z}$, we have

$$R_{n,1}(z) - \bar{\partial} C_n(z) = nr_n^2 \Delta Q(r_n z) + \Delta \log R_{n,1}(z)$$

and the equality holds pointwise. The canonical decomposition of $Q = Q_0 + \operatorname{Re} H + Q_1$ shows that

$$nr_n^2 \Delta Q(r_n z) = nr_n^{2d} \Delta Q_0(z) + O(nr_n^{2d+1})$$

holds uniformly on each compact subset of $\mathbb{C}$ as $n \to \infty$. Since $nr_n^{2d} \to 1$ as $n \to \infty$, we obtain (3.5.5). $\square$

Remark 3.5.3. We here remark that the rescaled Ward's equation (3.5.5) can be written in terms of L_n as follows:

$$\bar{\partial} C_n(z) = L_n(z,z)\, e^{-Q_0(z)}(1+o(1)) + \Delta_z \log L_n(z,z) + o(1),$$

where $o(1) \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. This is because

$$R_n(z) = K_n(z,z) = L_n(z,z) e^{-Q_0(z) - Q_{n,1}(z)}$$

holds and $Q_{n,1}(z) = O(r_n)$ uniformly on compact subsets as $n \to \infty$.

3.5.2 The proof of Theorem 3.2.2 and Theorem 3.2.3

Lemma 3.5.4. *Suppose that $L(z,z)$ is not identically zero. If $L(z_0,z_0) = 0$, then $L(z,z) = |z-z_0|^2 \tilde{L}(z,z)$ for some Hermitian-entire $\tilde{L}$ and the zero z_0 is isolated. In this case, $z \mapsto \tilde{L}(z,z)$ is logarithmic subharmonic.*

Proof. Suppose that $L(z_0,z_0) = 0$ for some z_0. Then the mass one inequality

(3.3.3) implies that

$$\int_{\mathbb{C}} |L(z_0, w)|^2 \, e^{-Q_0(w)} \, dA(w) \leq L(z_0, z_0) = 0$$

and we obtain $L(z_0, w) = 0$ for all $w \in \mathbb{C}$. Since L is Hermitian, $L(z, z_0) = 0$ for all $z \in \mathbb{C}$. Since L is entire in z and $\bar{w}$, there exists a Hermitian-entire function $\tilde{L}$ such that

$$L(z, w) = (z - z_0)\overline{(w - z_0)} \, \tilde{L}(z, w), \quad \text{for} \quad z, w \in \mathbb{C}.$$

We assume that z_0 is not isolated, i.e., there exists a sequence $\{z_j\}_1^\infty$ of distinct zeros of R which converges to z_0. For each j, $L(z_j, w) = 0$ for all $w \in \mathbb{C}$ and hence, for fixed w, $L(z, w) = 0$ for all $z \in \mathbb{C}$ since $L(z, w)$ is analytic in z. This implies that L is identically zero, which is contrary to the assumption. Therefore, the zero z_0 is isolated.

For the second statement, we recall that the function $z \mapsto \log L(z, z)$ is subharmonic by Lemma 3.4.5. Thus it holds that

$$\Delta_z \log L(z, z) = \Delta_z \log \tilde{L}(z, z) + \delta_{z_0} \geq 0$$

in the sense of distributions. Since the zero z_0 is isolated, we can take a small disc $D = D(z_0, \epsilon)$ such that $L(z, z) > 0$ for all $z \in D \setminus \{z_0\}$. Then $\Delta_z \log \tilde{L}(z, z) \geq 0$ in the sense of distributions on $D(z_0, \epsilon) \setminus \{z_0\}$. We extend $\log \tilde{L}(z, z)$ analytically to z_0 if $\tilde{L}(z_0, z_0) > 0$. Then we have $\Delta_z \log \tilde{L}(z, z) \geq 0$ for $z \in D$. On the other hand, we define $\log \tilde{L}(z_0, z_0) = -\infty$ if $\tilde{L}(z_0, z_0) = 0$. In this case, we also have $\log \tilde{L}(z, z)$ is subharmonic at z_0. □

We now prove the Ward's equation (3.5.3) from the rescaled version of Ward's equation (3.5.5). For a holomorphic kernel L, we write

$$\lim_{l \to \infty} L_{n_l} = L$$

and recall the definitions of Berezin kernels B_n, B and their Cauchy transforms C_n, C given in (3.5.1) and (3.5.2).

Lemma 3.5.5. *Suppose that $L(z, z)$ does not vanish identically, and let $\mathcal{Z}$ be the set of isolated zeros of $L(z, z)$. Then $B_{n_l}(z, w) \to B(z, w)$ for all $z \in \mathbb{C} \setminus \mathcal{Z}$, $w \in \mathbb{C}$, and $C_{n_l} \to C$ uniformly on compact subsets of $\mathbb{C} \setminus \mathcal{Z}$.*

Moreover, C is bounded on $(\mathbb{C} \setminus \mathcal{Z}) \cap V$ for each compact subset V of $\mathbb{C}$.

Proof. With some cocycle function c_{n_l}, we have the convergence $c_{n_l} K_{n_l} \to K$ which is locally uniform on $\mathbb{C}^2$. Note that $K(z,z) > 0$ at $z \notin \mathcal{Z}$ and $B(z,w)$ is defined at $z \notin \mathcal{Z}$, $w \in \mathbb{C}$. Fix a number ϵ with $0 < \epsilon < 1$. There exists N such that if $l \geq N$, then

$$|B_{n_l}(z,w) - B(z,w)| < \epsilon^2$$

for all z, w with $|z| \leq 1/\epsilon$, $|w| \leq 2/\epsilon$, and $\operatorname{dist}(z, \mathcal{Z}) \geq \epsilon$. Thus for z with $|z| \leq 1/\epsilon$, $\operatorname{dist}(z, \mathcal{Z}) \geq \epsilon$, and $l \geq N$,

$$\int_{|z-w|<\frac{1}{\epsilon}} \left| \frac{B_{n_l}(z,w) - B(z,w)}{z-w} \right| dA(w) \leq \epsilon^2 \int_{|z-w|<\frac{1}{\epsilon}} \frac{1}{|z-w|} dA(w) = 2\epsilon. \tag{3.5.7}$$

On the other hand, since the mass-one inequality (3.3.3) implies that $\int B(z,w)\, dA(w) \leq 1$, we have

$$\int_{|z-w|<1/\epsilon} \left| \frac{B_{n_l}(z,w) - B(z,w)}{z-w} \right| dA(w) \leq 2\epsilon. \tag{3.5.8}$$

By (3.5.7) and (3.5.8), $C_{n_l} \to C$ uniformly on compact subsets of $\mathbb{C} \setminus \mathcal{Z}$.

For the second statement, we fix a compact subset V of $\mathbb{C}$. By Lemma 3.3.2, there exists $M = M_V$ such that for all z, w with $z \in V \setminus \mathcal{Z}$ and $\operatorname{dist}(w, V) < 1$,

$$B_{n_l}(z,w) = \frac{|K_{n_l}(z,w)|^2}{K_{n_l}(z,z)} \leq K_{n_l}(w,w) \leq M.$$

Now we divide the integration region of C_{n_l} into two parts as follows: for $z \in V \setminus \mathcal{Z}$,

$$\begin{aligned} |C_{n_l}(z)| &\leq \int_{|z-w|<1} \left| \frac{B_{n_l}(z,w)}{z-w} \right| dA(w) + \int_{|z-w|>1} \left| \frac{B_{n_l}(z,w)}{z-w} \right| dA(w) \\ &\leq M \int_{|z-w|<1} \frac{1}{|z-w|} dA(w) + \int B_{n_l}(z,w) dA(w) \leq 2M + 1. \end{aligned}$$

We conclude that $C = \lim C_{n_l}$ also satisfies that $|C(z)| \leq 2M + 1$ for all $z \in V \setminus \mathcal{Z}$. $\square$

Lemma 3.5.6. *Suppose that $L(z,z)$ does not vanish identically. Then the Ward's equation*

$$\bar{\partial}C(z) = L(z,z)\,e^{-Q_0(z)} - \Delta_z \log L(z,z) \tag{3.5.9}$$

holds in the sense of distributions.

Proof. By Lemma 3.5.2, we have

$$\bar{\partial}C_n(z) = L_n(z)\,e^{-Q_0(z)}(1+o(1)) + \Delta \log L_n(z,z) + o(1),$$

where $o(1) \to 0$ uniformly on compact subsets of $\mathbb{C}$. Here, by the above lemma, $C_{n_l} \to C$ locally uniformly on $\mathbb{C} \setminus \mathcal{Z}$. Since for each compact set V, C_{n_l} and C are bounded on $V \setminus \mathcal{Z}$ and $V \cap \mathcal{Z}$ is a finite set, we have $C_{n_l} \to C$ and $\bar{\partial}C_{n_l} \to \bar{\partial}C$ in the sense of distributions. On the right-hand side, $\Delta_z \log L_{n_l}(z,z) \to \Delta_z \log L(z,z)$ in the sense of distributions. Hence, the lemma is proved. $\square$

Proof of Lemma 3.5.1. Lemma 3.5.1 consists of the following two statements. The first one is zero-one law which states that if $L(z,z)$ does not vanish identically, then $L(z,z)$ does not have any zero. The second one is that the rescaled Ward's equation (3.5.3),

$$\bar{\partial}C(z) = L(z,z)\,e^{-Q_0(z)} - \Delta_z \log L(z,z),$$

holds for all $z \in \mathbb{C}$.

First, we assume that $L(z,z)$ has a zero at $z = z_0$. We take a small disk D centered at z_0 such that $L(z,z) > 0$ for all $z \in D \setminus \{z_0\}$. Now we write $L(z,z) = |z-z_0|^2\,\tilde{L}(z,z)$ and consider the measures

$$d\nu(z) = \chi_D(z) \cdot \Delta_z \log L(z,z)\,dA(z),$$
$$d\tilde{\nu}(z) = \chi_D(z) \cdot \Delta_z \log \tilde{L}(z,z)\,dA(z).$$

Since $\log L(z,z)$ and $\log \tilde{L}(z,z)$ are subharmonic by Lemma 3.4.5 and Lemma 3.5.4, these measures are positive and $\nu = \delta_{z_0} + \tilde{\nu}$. Writing

$$C^{\nu}(z) = \int_{\mathbb{C}} \frac{1}{z-w}\,d\nu(w)$$

for the Cauchy transform of ν, we have $C^{\nu}(z) = \frac{1}{z-z_0} + C^{\tilde{\nu}}(z)$ for $z \in D$. By

Ward's equation (3.5.9), we have that

$$L(z,z)\,e^{-Q_0(z)} = \bar{\partial}C(z) + \bar{\partial}C^{\nu}(z)$$

in the sense of distributions on D. We refer to Weyl's lemma which implies that if a distribution u satisfies $\Delta u = f$ for some smooth f, then u is also smooth. (See [26], p.110.) Hence, we have

$$v(z) = C(z) + C^{\nu}(z) = C(z) + C^{\tilde{\nu}}(z) + \frac{1}{z-z_0}, \quad z \in D$$

for some smooth function v defined in D. Since C is bounded in $D \setminus \{z_0\}$ by Lemma 3.5.5, $C^{\nu}(z)$ must be bounded as $z \to z_0$. However, this implies that ν has no mass at z_0 and $\tilde{\nu} = -\delta_{z_0} + \nu$, which contradicts that $\tilde{\nu}$ is positive. Therefore, we conclude that $L(z,z)$ does not have any zeros, which proves the first statement.

Since $L(z,z)$ is positive everywhere, $\log L(z,z)$ is smooth on $\mathbb{C}$. Thus, the right-hand side of Ward's equation (3.5.9) is smooth. By Weyl's lemma again, C is smooth everywhere and hence, the equation (3.5.9) holds pointwise on $\mathbb{C}$. The proof of the second statement is complete. $\square$

3.6 Dominant radial singularities

In this section, we prove Theorem 3.2.5. Recall the canonical decomposition $Q = Q_0 + \operatorname{Re} H + Q_1$ and suppose that $Q_0(z) = Q_0(|z|)$. Now we fix a rotationally symmetric holomorphic kernel L, i.e., $L(z,w) = L(z\,e^{it}, w\,e^{it})$ for all $t \in \mathbb{R}$. We first take a look at the following lemma.

Lemma 3.6.1. *A Hermitian-entire function L is rotationally symmetric if and only if $L(z,w) = E(z\bar{w})$ for some entire function E.*

Proof. Suppose that L is rotationally symmetric. Since $L(z,w)$ is Hermitian-entire, L can be written in the form

$$L(z,w) = \sum_{j,k=0}^{\infty} a_{j,k}\, z^j\, \bar{w}^k,$$

where $\alpha_{j,k}$ are complex numbers with $a_{k,j} = \overline{a_{j,k}}$. Noting that for all $t \in \mathbb{R}$

$$L(z\,e^{it}, w\,e^{it}) = \sum_{j,k=0}^{\infty} a_{j,k}\, e^{i(j-k)t}\, z^j \bar{w}^k = \sum_{j,k=0}^{\infty} a_{j,k}\, z^j \bar{w}^k,$$

it is clear that $a_{j,k} = 0$ if $j \neq k$. Thus we obtain $L(z, w) = E(z\bar{w})$ where $E(z) = \sum_{j=0}^{\infty} a_{j,j}\, z^j$. The converse is obvious. □

We write $E(z) = \sum_{j=0}^{\infty} a_j z^j$, $a_j \in \mathbb{R}$ and $d\mu_0 = e^{-Q_0}\, dA$. Recall that L satisfies the following mass-one inequality (See Lemma 3.3.4.):

$$\int_{\mathbb{C}} |L(z, w)|^2\, d\mu_0(w) \leq L(z, z). \tag{3.6.1}$$

The mass-one inequality gives a relation among the coefficients a_j.

Lemma 3.6.2. *The mass-one inequality* (3.6.1) *is equivalent to the following condition:*

$$\sum |a_j|^2\, |z|^{2j}\, \| \, w^j \, \|^2_{L^2(\mu_0)} \leq \sum a_j\, |z|^{2j}\,. \tag{3.6.2}$$

Proof. Writing that $L(z, w) = \sum a_j (z\bar{w})^j$, the proof is straightforward. □

Together with the mass-one inequality, the Ward's equation

$$\bar{\partial} C(z) = L(z, z)\, e^{-Q_0(z)} - \Delta_z \log L(z, z) \tag{3.6.3}$$

in Section 3.5 is used to prove the universality result. We first need an elementary lemma in [4] to calculate the Cauchy transform on the left-hand side of (3.6.3).

Lemma 3.6.3. *For $k \in \mathbb{Z}$, the following calculation holds:*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ik\theta}}{z - e^{i\theta}}\, d\theta = \begin{cases} -z^{k-1} & \text{if } |z| < 1,\ k \geq 1, \\ \ \ z^{k-1} & \text{if } |z| > 1,\ k \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6.4}$$

Proof. We note that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ik\theta}}{z - e^{i\theta}}\, d\theta = \frac{1}{2\pi i} \int_{|w|=1} \frac{w^{k-1}}{z - w}\, dw.$$

For $k \geq 1$, by Cauchy integral formula, we have

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{w^{k-1}}{z-w}\,dw = \begin{cases} -z^{k-1} & \text{if } |z|<1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $k \leq 0$ and $|z| < 1$, then by the residue theorem

$$\begin{aligned} \frac{1}{2\pi i}\int_{|w|=1}\frac{w^{k-1}}{z-w}\,dw &= \mathrm{Res}_{w=z}\frac{1}{w^{1-k}(z-w)} + \mathrm{Res}_{w=0}\frac{1}{w^{1-k}(z-w)} \\ &= -z^{k-1} + z^{k-1} = 0. \end{aligned}$$

If $k \leq 0$ and $|z| > 1$, by the residue theorem again,

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{w^{k-1}}{z-w}\,dw = \mathrm{Res}_{w=0}\frac{1}{w^{1-k}(z-w)} = z^{k-1}.$$

Hence, the proof is complete. □

Lemma 3.6.4. *The Ward's equation* (3.6.3) *is equivalent to the following conditions: For each integer $k \geq 1$,*

$$a_k = 0 \quad or \quad \sum_{j=0}^{k-1} a_j \parallel z^j \parallel^2_{L^2(\mu_0)} = k. \tag{3.6.5}$$

Proof. We first compute the Cauchy transform $C(z)$ in (3.6.3) as follows:

$$\begin{aligned} C(z) &= \frac{1}{L(z,z)}\int_{\mathbb{C}}\frac{|L(z,w)|^2}{z-w}e^{-Q_0(w)}\,dA(w) \\ &= \frac{1}{E(|z|^2)}\sum_{j,k} a_j\bar{a}_k z^j\bar{z}^k\int_{\mathbb{C}}\frac{\bar{w}^j w^k}{z-w}e^{-Q_0(w)}\,dA(w). \end{aligned}$$

Using (3.6.4), we have

$$\begin{aligned} C(z) =& \frac{2}{E(|z|^2)}\sum_{j,k} a_j\bar{a}_k z^j\bar{z}^k\int_0^\infty r^{k+n}e^{-Q_0(r)}dr\int_0^{2\pi}\frac{e^{i(k-j)\theta}}{z/r-e^{i\theta}}\frac{d\theta}{2\pi} \\ =& \frac{2}{E(|z|^2)}\sum_{j,k} a_j\bar{a}_k z^j\bar{z}^k\,(A^1_{j,k}(z) - A^2_{j,k}(z)), \end{aligned}$$

where

$$A^1_{j,k}(z) = \int_0^{|z|} e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \chi(k \le j)\, dr,$$
$$A^2_{j,k}(z) = \int_{|z|}^{\infty} e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \chi(k \ge j+1)\, dr.$$

If we write

$$S_1(z) = \frac{2}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \int_0^{|z|} e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} dr,$$
$$S_2(z) = \frac{2}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \int_0^{\infty} e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \chi(k \ge j+1)\, dr,$$

then we have $C(z) = S_1(z) - S_2(z)$. Now, S_1 is computed as follows:

$$\begin{aligned} S_1(z) &= \frac{2}{zE(|z|^2)} \sum_{j,k} a_j \bar{a}_k \, |z|^{2k} \int_0^{|z|} e^{-Q_0(r)} r^{2j+1}\, dr \\ &= \frac{1}{z} \int_0^{|z|^2} e^{-Q_0(\sqrt{s})} E(s)\, ds, \end{aligned}$$

whence we obtain

$$\bar{\partial} S_1(z) = e^{-Q_0(z)} E(|z|^2) = e^{-Q_0(z)} L(z,z).$$

On the other hand, S_2 can be written in the form

$$\begin{aligned} S_2(z) &= \frac{2}{E(|z|^2)} \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} a_j \bar{a}_k z^{k-1} \bar{z}^k \int_0^{\infty} e^{-Q_0(r)} r^{2j+1}\, dr \\ &= \frac{1}{E(|z|^2)} \sum_{k=1}^{\infty} \bar{a}_k z^{k-1} \bar{z}^k \sum_{j=0}^{k-1} a_j \left\| z^j \right\|^2_{L^2(\mu_0)}. \end{aligned}$$

Now we return to the Ward's equation (3.6.3). By the above computations, the Ward's equation is equivalent to $\bar{\partial} S_2(z) = \Delta_z \log L(z,z)$ for all $z \in \mathbb{C}$, which is equivalent to that $S_2(z) - \partial_z \log L(z,z)$ is entire. Write $F(z) :=$

$S_2(z) - \partial_z \log L(z,z)$. Since a direct calculation gives

$$S_2(z) - \partial_z \log L(z,z) = \frac{1}{E(|z|^2)} \sum_{k=1}^{\infty} \bar{a}_k z^{k-1} \bar{z}^k \left(\sum_{j=0}^{k-1} a_j \parallel z^j \parallel_{L^2(\mu_0)}^2 - k \right),$$

we obtain

$$F(z) \cdot \sum_{j=0}^{\infty} a_j |z|^{2j} = \sum_{k=1}^{\infty} \bar{a}_k \left(\sum_{j=0}^{k-1} a_j \parallel z^j \parallel_{L^2(\mu_0)}^2 - k \right) z^{k-1} \bar{z}^k. \qquad (3.6.6)$$

If F is entire, then F has a holomorphic Taylor expansion at 0. Hence, by comparing coefficients, we can conclude that F is entire if and only if for each $k \geq 1$

$$a_k = 0 \quad \text{or} \quad \sum_{j=0}^{k-1} a_j \parallel z^j \parallel_{L^2(\mu_0)}^2 = k.$$

The proof is complete. □

We here need the estimate of the limiting correlation function R, which is proved in Section 5. In terms of E, the estimate can be written as

$$E(|z|^2) = \Delta Q_0(z)\, e^{Q_0(z)}(1 + o(1)) \quad \text{as} \quad z \to \infty. \qquad (3.6.7)$$

Recall that $L_a^2(\mu_0)$ is the Bergman space consisting of all holomorphic functions in $L^2(\mu_0)$. The following Lemma proves Theorem 3.2.5.

Lemma 3.6.5. *With the growth estimate* (3.6.7)*, we have*

$$E(z) = \sum_{j=0}^{\infty} \frac{1}{\parallel z^j \parallel_{L^2(\mu_0)}^2} z^j,$$

and hence, $L = L_0$ where L_0 is the Bergman kernel of the space $L_a^2(\mu_0)$.

Proof. Clearly by the estimate (3.6.7), E is non-trivial. Then E does not have any zero by the zero-one law, Theorem 3.2.2, so that we have $a_0 \neq 0$. In addition, the mass-one inequality (3.6.1) implies that

$$0 < a_0 \leq 1/ \parallel 1 \parallel_{L^2(\mu_0)}^2 .$$

We claim that for all k, $a_k = 1/ \parallel z^k \parallel_{L^2(\mu_0)}^2$. We want to show that it is not the case that $a_k = 0$ in (3.6.5). Note that since E satisfies the growth estimate (3.6.7), E is not a polynomial. It means that for any $k \geq 0$, there exists $N \in \mathbb{N}$ such that $k < N$ and $a_N \neq 0$.

To the contrary, we assume that $a_1 = 0$. Then there exists a number $N_1 > 1$ such that $a_j = 0$ for all $1 \leq j \leq N_1 - 1$ but $a_{N_1} \neq 0$. By (3.6.5), this implies

$$a_0 \| 1 \|^2_{L^2(\mu_0)} = \sum_{j=0}^{N_1-1} a_j \| z^j \|^2_{L^2(\mu_0)} = N_1,$$

which is a contradiction. Hence, $a_1 \neq 0$ and, by (3.6.5) again, we obtain $a_0 = 1/ \| 1 \|^2_{L^2(\mu_0)}$. Applying this argument inductively, we have for all k,

$$a_k = \frac{1}{\| z^k \|^2_{L^2(\mu_0)}}.$$

Hence, we conclude that

$$L(z,w) = E(z\bar{w}) = \sum_{j=0}^{\infty} \frac{(z\bar{w})^j}{\| z^j \|^2_{L^2(\mu_0)}}.$$

On the other hand, the Bergman kernel L_0 can be written in the form

$$L_0(z,w) = \sum_{j=0}^{\infty} \phi_j(z)\bar{\phi}_j(w),$$

where ϕ_j is the j-th orthonormal polynomial with respect to the measure μ_0. Since Q_0 is radially symmetric, we obtain $\phi_j(z) = z^j / \left\| z^j \right\|_{L^2(\mu_0)}$. Therefore, $L = L_0$. $\square$

3.7 Homogeneous singularities

In this section, we prove Theorem 3.2.6. Let Q be the potential which has a homogeneous singularity of type $2d-2$ at 0. Then Q is a homogeneous polynomial of degree $2d$ and the canonical decomposition of Q is given by

$$Q(\zeta) = Q_0(\zeta) + \operatorname{Re} H(\zeta), \quad H(\zeta) = \alpha\, \zeta^{2d}.$$

Write $d\mu_0 = e^{-Q_0}\, dA$, and let L_n be the rescaled holomorphic kernel defined in Section 3.3.2 by

$$L_n(z, w) = k_n(z, w)\, e^{-n(H(r_n z)+\bar{H}(r_n w))/2},$$

where $r_n = n^{-1/2d}$ is the microscopic scale. In this setting, we have

$$L_n(z, w) = k_n(z, w)\, e^{-(H(z)+\bar{H}(w))/2}$$

and L_n can be regarded as the reproducing kernel for the space $\mathcal{H}_n$ of holomorphic functions defined by

$$\mathcal{H}_n = \{p(z) \cdot e^{-H(z)/2} \,:\, p \in \mathrm{Pol}(n)\}$$

equipped with the norm of $L^2(\mu_0)$. Here $\mathrm{Pol}(n)$ is the space of analytic polynomials of degree less than n.

Lemma 3.7.1. *L_n converges to L_0 uniformly on compact subsets of $\mathbb{C}^2$ as $n \to \infty$, where L_0 is the Bergman kernel of $L^2_a(\mu_0)$.*

Proof. By Lemma 3.3.3, every subsequence of $\{L_n\}$ has a subsequence which converges locally uniformly on $\mathbb{C}^2$. Since $\mathcal{H}_n$ is increasing and the inclusions $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ are isometric, $L_{n+1} - L_n$ is a positive matrix. This implies that for each z, the sequence $\{L_n(z, z)\}$ is increasing, whence there exists a unique limiting kernel L such that $\lim L_n = L$.

Now we claim that $L = L_0$. For every polynomial p, we have

$$p(z) = \lim_{n\to\infty} \int p(w) L_n(z, w) d\mu_0(w) = \int p(w) L(z, w)\, d\mu_0(w).$$

Since polynomials are dense in $L^2_a(\mu_0)$,

$$f(z) = \int f(w) L(z, w)\, dA(w)$$

for all $f \in L^2_a(\mu_0)$, which implies that L is the reproducing kernel for $L^2_a(\mu_0)$. The proof is complete. □

Chapter 4

Conical singularities

In this chapter, we study the random normal matrix model with conical singularities in the bulk. We prove the existence of the scaling limit of eigenvalue point processes at a conical singularity. we also obtain a universality result for homogeneous potentials. This chapter is based on [7].

4.1 Introduction and results

4.1.1 Perturbation of potentials

Consider a potential Q which is admissible and real analytic in the interior of the set $\{Q < \infty\}$. Let p be a point in the interior of the droplet S_Q. Since Q is real analytic near p, we obtain the Taylor expansion of ΔQ about p as

$$\Delta Q(\zeta) = P(\zeta - p) + O(|\zeta - p|^{2d-1}),$$

where $P(x + iy)$ is a homogeneous polynomial in x, y of degree $2d - 2$ for some integer $d \geq 1$. We assume that P is positive definite, i.e., $P(\zeta) > 0$ for all $\zeta \neq 0$. If $d = 1$, we say Q is *regular* at p, and if $d \geq 2$, we say Q has a *bulk singularity of type* $2d - 2$ at p. (cf. Definition 3.1.1.)

For a fixed real parameter $c > -1$, we define a perturbed potential V_n by

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n} \log |\zeta - p| \,.$$

Then, we say V_n has a **conical singularity** of order c at p.

Let $\{\zeta_j\}_{j=1}^n$ be the eigenvalue system of the random normal matrix ensemble associated with V_n. The joint distribution of the eigenvalues $\{\zeta_j\}_1^n$ takes the form

$$d\mathbf{P}_n(\boldsymbol{\zeta}) = \frac{1}{Z_n} \prod_{1\le j<k\le n} |\zeta_j - \zeta_k|^2 \, e^{-n\sum_{j=1}^n V_n(\zeta_j)} dA^{\otimes n}(\boldsymbol{\zeta}), \tag{4.1.1}$$

where $\boldsymbol{\zeta} = (\zeta_j)_{j=1}^n \in \mathbb{C}^n$, $dA^{\otimes n}(\boldsymbol{\zeta}) = dA(\zeta_1)\cdots dA(\zeta_n)$, and Z_n is the normalizing constant which makes $\mathbf{P}_n$ a probability measure.

First, it is natural to examine how the droplet changes after a perturbation of potential. Let σ_n be the first marginal measure of the system $\{\zeta_j\}_1^n$, which is the expectation of the empirical measure $\frac{1}{n}\sum_{j=1}^n \delta_{\zeta_j}$ of the eigenvalues. As discussed in Section 2.3.2, it is well-known that the sequence of marginal measures of the system associated with the (original) potential Q converges to the equilibrium measure σ_Q in the weak-star sense of measures as n goes to ∞. The following proposition asserts that this convergence to σ_Q also holds true for the case of the perturbed potential V_n.

Proposition 4.1.1. *Let σ_n be the marginal measure of the eigenvalue system associated with V_n. We have as $n \to \infty$,*

$$\sigma_n \to \sigma_Q$$

in the weak-star sense of measures.

Thus, we can deduce from Proposition 4.1.1 that as n tends to infinity, the eigenvalues still accumulate on the droplet S_Q and obey the equilibrium measure σ_Q even when the associated potential V_n has a logarithmic singularity. We give a proof of Proposition 4.1.1 in Section 4.4.

4.1.2 Conical singularities

We express the joint distribution (4.1.1) of the eigenvalue system $\{\zeta_j\}_1^n$ in the form

$$d\mathbf{P}_n(\boldsymbol{\zeta}) = \frac{1}{Z_n} e^{-nH_n(\boldsymbol{\zeta})} dA^{\otimes n}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} = (\zeta_j) \in \mathbb{C}^n,$$

where H_n is given by

$$H_n(\boldsymbol{\zeta}) = -\sum_{j\neq k} \log|\zeta_j - \zeta_k| + \sum_{j=1}^{n} nV_n(\zeta_j)$$
$$= -\sum_{j\neq k} \log|\zeta_j - \zeta_k| + \sum_{j=1}^{n} (nQ(\zeta_j) - 2c\log|\zeta_j - p|)\,.$$

Here, H is the Hamiltonian of the system consisting of n particles with charge $+1$ and an extra particle with charge $+c$ at p, influenced by the external field nQ. If c is positive, there exists repulsion between p and other particle ζ_j. On the other hand, if c is negative, there is attraction between p and ζ_j.

Another interpretation of this model can be found in the literature [12] and [37]. The weight $e^{-nV_n(z)} = |z-p|^{2c}\, e^{-nQ(z)}$ is related to the conformal metric on the Riemann surface with a conical singularity.

From now on, we assume that $p = 0$ without loss of generality.

4.1.3 Microscopic scale

Let V_n have a conical singularity of order c at the origin, and write

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n}\log|\zeta|\,. \tag{4.1.2}$$

Then, Q has the canonical decomposition (cf. Section 3.1.3),

$$Q = Q_0 + \operatorname{Re} H + Q_1, \tag{4.1.3}$$

where Q_0 is homogeneous of degree $2d$, $Q_1 = O(|\zeta|^{2d+1})$ as $\zeta \to 0$, and

$$H(\zeta) = Q(0) + 2\partial Q(0)\cdot\zeta + \cdots + \frac{2}{(2d)!}\partial^{2d}Q(0)\cdot\zeta^{2d}.$$

As discussed in the Section 3, we define the microscopic scale r_n by the equation

$$n\int_{D(0,r_n)} \Delta Q\, dA = 1 + c. \tag{4.1.4}$$

Then, we see that r_n satisfies

$$r_n = \tau_0(1+c)^{1/2d} n^{-1/2d}(1+o(1)) \qquad as \quad n\to\infty, \tag{4.1.5}$$

where τ_0 is the *modulus* at 0, i.e.,

$$\tau_0^{-2d} = \frac{1}{2\pi d}\int_0^{2\pi} \Delta Q_0(e^{i\theta})\,d\theta.$$

Indeed, since for $z = re^{i\theta} \in D(0, r_n)$

$$\Delta Q(re^{i\theta}) = r^{2d-2}\Delta Q_0(e^{i\theta}) + O(r_n^{2d-1}),$$

we obtain from the equation (4.1.4)

$$1 + c = n\int_{D(0,r_n)} r^{2d-2}\Delta Q_0(e^{i\theta})\,dA + O(r_n^{2d+1}),$$

which implies (4.1.5). From now on, we write $\tau_c = \tau_0(1+c)^{1/2d}$.

4.1.4 Main results

Let V_n be a potential which has a conical singularity of order c at 0, i.e.,

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n}\,|\zeta|\,.$$

Consider the measure $d\mu_n(\zeta) = e^{-nV_n(\zeta)}\,dA(\zeta)$. Let $\mathcal{P}_n(\mu_n)$ be the space of all holomorphic polynomials of degree at most $n-1$, equipped with the norm

$$\|\, p\,\|^2_{L^2(\mu_n)} = \int_{\mathbb{C}} |p|^2\,d\mu_n = \int_{\mathbb{C}} |p(\zeta)|^2\,|\zeta|^{2c}\,e^{-nQ(\zeta)}\,dA(\zeta).$$

A system of eigenvalues $\{\zeta_j\}_1^n$ associated with the potential V_n forms a determinantal point process with the correlation kernel

$$\mathbf{K}_n(\zeta,\eta) = \mathbf{k}_n(\zeta,\eta)\,e^{-nV_n(\zeta)/2-nV_n(\eta)/2}$$

where $\mathbf{k}_n$ is the reproducing kernel of $\mathcal{P}_n(\mu_n)$.

Let $\Theta_n = \{z_j\}_{j=1}^n$ be the rescaled process at 0 defined by

$$z_j = r_n^{-1}\zeta_j \quad \text{for} \quad j = 1,\cdots,n.$$

Then, Θ_n forms a determinantal point process with the correlation kernel

$$K_n(z,w) = r_n^2\,\mathbf{K}_n(\zeta,\eta), \quad z = r_n^{-1}\zeta,\; w = r_n^{-1}\eta.$$

Existence of limiting kernels

We write a number $c > -1$ as $c = q + c'$ where q is a non-negative integer and $-1 < c' \leq 0$, and we put

$$V_0(z) = Q_0(z) - 2c \log|z|, \tag{4.1.6}$$

where Q_0 is the dominant part in the canonical decomposition (4.1.3).

Theorem 4.1.2. *There exists a sequence of cocycles c_n such that every subsequence of $\{c_n K_n\}$ has a further subsequence which is convergent for all z, $w \in \mathbb{C} \setminus \{0\}$. Moreover, the limit point K is of the form*

$$K(z,w) = L(z,w)\, e^{-V_0(\tau_c z)/2 - V_0(\tau_c w)/2},$$

where $l(z,w) = (z\bar{w})^q L(z,w)$ is Hermitian-entire.

Remark 4.1.3. More precisely, the convergence $c_{n_k} K_{n_k} \to K$ in Theorem 4.1.2 is uniform in the sense that as $k \to \infty$,

$$|z\bar{w}|^{-c'}(c_{n_k} K_{n_k})(z,w) \to |z\bar{w}|^{-c'} K(z,w)$$

uniformly on each compact subset of $\mathbb{C}^2$.

Universality result for homogeneous potentials

Assume that $Q(\zeta)$ is a homogeneous polynomial in ζ, $\bar{\zeta}$ of degree $2d$. By the canonical decomposition, $Q(\zeta) = Q_0(\zeta) + \alpha \operatorname{Re} \zeta^{2d}$ for $\alpha \in \mathbb{C}$.

Write $d\mu_0(z) = e^{-V_0(\tau_c z)}\, dA(z)$, where V_0 is as above in (4.1.6). We consider the Bergman space $L_a^2(\mu_0)$ of all entire functions u such that $\| u \|_{L^2(\mu_0)} < \infty$, where

$$\| u \|_{L^2(\mu_0)}^2 = \int_{\mathbb{C}} |u|^2 \, d\mu_0.$$

Let L_0 be the Bergman kernel of the Bergman space $L_a^2(\mu_0)$.

Theorem 4.1.4. *Suppose that Q is a homogeneous polynomial of degree $2d$. Then, as $n \to \infty$, the rescaled system Θ_n converges to the point process with the correlation kernel*

$$K(z,w) = L_0(z,w)\, e^{-V_0(\tau_c z)/2 - V_0(\tau_c w)/2}.$$

4.1.5 Example : Mittag-Leffler ensembles.

Consider the case when $Q(\zeta) = |\zeta|^{2d}$. For $c > -1$, put

$$V_n(\zeta) = |\zeta|^{2d} - \frac{2c}{n} \log |\zeta|$$

and consider the system $\{\zeta_j\}_1^n$ of eigenvalues associated with the perturbed potential V_n. We rescale the system at 0 by $z_j = r_n^{-1}\zeta_j$ where the microscopic scale r_n is given by $r_n = \tau_c n^{-1/2d}$ with $\tau_c = ((1+c)/d)^{1/2d}$. Then the correlation kernel K_n of the rescaled system $\{z_j\}_1^j$ is given by

$$K_n(z,w) = d\tau_c^2 \sum_{j=0}^{n-1} \frac{(\tau_c^2 z\bar{w})^j}{\Gamma\left(\frac{j+1+c}{d}\right)} \cdot \left|\tau_c^2 z\bar{w}\right|^{2c} e^{-|\tau_c z|^{2d}/2 - |\tau_c w|^{2d}/2},$$

where Γ is the Gamma function. We write $V_0(z) = |z|^{2d} - 2c \log |z|$ and obtain that $\lim_{n\to\infty} K_n = K$ where

$$K(z,w) = d\tau_c^2 \cdot E_{1/d,(1+c)/d}(\tau_c^2 z\bar{w})\, e^{-V_0(\tau_c z)/2 - V_0(\tau_c w)/2}.$$

Here $E_{a,b}$ is the two-parametric Mittag-Leffler function

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}.$$

For more detail about the function $E_{a,b}$ we refer to [24]. From the asymptotic estimate for Mittag-Leffler function in [24], we obtain that

$$R(z) = K(z,z) = \tau_c^{2d} d^2 \,|z|^{2d-2} + O(|z|^{-2-2c}\, e^{-|\tau_c z|^{2d}}) \quad \text{as} \quad z \to \infty.$$

For the asymptotic estimate for the limiting one point function in the general case, see Chapter 5.

4.2 Existence of limiting kernels

In this section, we prove Theorem 4.1.2 by using the estimate for the reproducing kernels and a normal families argument (cf. Chapter 3.3).

Let $V_n(z) = Q(z) - (2c/n) \log |z|$ be a potential with a conical singularity of order c at 0. Recall that $Q = Q_0 + \operatorname{Re} H + Q_1$ is the canonical decomposition of Q and Q_0 is a homogeneous polynomial of degree $2d$ for some d. By multiplying the potential by a suitable constant, we assume that the microscopic scale r_n is equal to $n^{-1/2d}$. Write $d\mu_n = e^{-nV_n}\, dA$.

4.2.1 Estimates for the reproducing kernels

Now let $\tilde{V}_n$ be the rescaled potential defined by

$$\tilde{V}_n(z) = nV_n(r_n z)$$

and $d\tilde{\mu}_n$ be the rescaled measure defined by $d\tilde{\mu}_n = e^{-\tilde{V}_n}\, dA$. We denote by $\mathcal{P}_n(\tilde{\mu}_n)$ the space of all homogeneous polynomials of degree at most $n-1$ equipped with the norm of $L^2(\tilde{\mu}_n)$. Then the rescaled kernel

$$k_n(z,w) = r_n^2 \mathbf{k}_n(\zeta,\eta), \quad \zeta = r_n z,\ \eta = r_n w,$$

is the reproducing kernel for $\mathcal{P}_n(\tilde{\mu}_n)$ where $\mathbf{k}_n$ is the reproducing kernel for $\mathcal{P}_n(\mu_n)$.

We first prove some a priori inequalities.

Lemma 4.2.1. *Suppose that $-1 < c \le 0$. Then for large T there is a constant $C = C(T)$ such that for all $u \in L_a^2(\tilde{\mu}_n)$ we have*

$$|u(z)|^2\, e^{-nQ(r_n z)} \le C r_n^{-2c} \,\|\, u \,\|^2_{L^2(\tilde{\mu}_n)}\,, \quad |z| \le T. \tag{4.2.1}$$

Moreover, the constant $C(T)$ satisfies

$$C(T) \le C_0 T^{4d-2c-2} \tag{4.2.2}$$

for some constant C_0 which is independent of T.

Proof. Fix a number δ with $0 < \delta < 1/2$. Then we choose α so that

$$\alpha > \sup\{\Delta Q_0(z) \ :\ |z| \le T + \delta\}. \tag{4.2.3}$$

Since $a_1\, |z|^{2d-2} \le \Delta Q_0(z) \le a_2\, |z|^{2d-2}$ for some $a_1,\, a_2 > 0$, α can be taken to be proportional to T^{2d-2}. Consider the function

$$F_n(z) = |u(z)|^2\ e^{-\tilde{V}_n(z)+\alpha|z|^2}.$$

For $0 < |z| \le T + \delta$, we have $\Delta \tilde{V}_n(z) < \alpha$ for sufficient large n since

$$\Delta \tilde{V}_n(z) = \Delta Q_0(z) + O(r_n), \quad n \to \infty,$$

where $r_n \sim n^{-1/2d}$. Thus we have for sufficiently large n

$$\Delta \log F_n(z) = \Delta \log |u(z)|^2 - \Delta \tilde{V}_n(z) + \alpha > 0,$$

which implies that F_n is subharmonic on $0 < |z| \le T + \delta$.

To see the inequality (4.2.1), we first assume $\delta \le |z| \le T$ and write

$$|u(z)|^2 \, e^{-\tilde{V}_n(z)+\alpha|z|^2} \le 4\delta^{-2} \int_{D(z,\delta/2)} |u(w)|^2 \, e^{-\tilde{V}_n(w)+\alpha|w|^2} \, dA(w). \quad (4.2.4)$$

Multiplying each side by $e^{-\alpha|z|^2 - 2c\log|r_n z|}$, we have for $\delta \le |z| \le T$

$$\begin{aligned} |u(z)|^2 \, e^{-nQ(r_n z)} &\le 4\delta^{-2} e^{\alpha(|z+\delta/2|^2 - |z|^2) - 2c\log|r_n z|} \int_{D(z,\delta/2)} |u|^2 \, e^{-\tilde{V}_n} \, dA \\ &\le 4\delta^{-2} r_n^{-2c} e^{\alpha\delta T + \alpha\delta^2/4 - 2c\log|T|} \, \| \, u \, \|^2_{L^2(\tilde{\mu}_n)} . \end{aligned}$$

Taking $\delta = T^{1-2d}$ and $\alpha \le bT^{2d-2}$ for some $b > 0$, we choose C_1 such that

$$|u(z)|^2 \, e^{-nQ(r_n z)} \le C_1 r_n^{-2c} \, \| \, u \, \|^2_{L^2(\tilde{\mu}_n)}$$

and

$$C_1 \le 4T^{-2+4d-2c} e^{b(1+T^{-2d}/4)}.$$

Now we suppose that $|z| < \delta$ with $\delta = T^{1-2d}$. Since the function

$$|u(z)|^2 \, e^{-nQ(r_n z) + \alpha|z|^2}$$

is subharmonic on $D(0, T+\delta)$ for sufficiently large n by (4.2.3), the sub-mean value property implies

$$|u(z)|^2 \, e^{-nQ(r_n z)+\alpha|z|^2} \le \delta^{-2} \int_{D(z,\delta)} |u(w)|^2 \, e^{-nQ(r_n w)+\alpha|w|^2} dA(w).$$

It follows that for $|z| < \delta$,

$$
\begin{aligned}
|u(z)|^2 e^{-nQ(r_n z)} &\le \delta^{-2} e^{2|z|\delta+\delta^2} \int_{D(z,\delta)} |u(w)|^2 \, e^{-nQ(r_n w)} dA(w) \\
&\le \delta^{-2} e^{\alpha(2|z|\delta+\delta^2)} \int_{D(z,\delta)} |u(w)|^2 \, e^{-nQ(r_n w)+2c\log|w|} dA(w) \\
&= \delta^{-2} e^{\alpha(2|z|\delta+\delta^2)} r_n^{-2c} \parallel u \parallel^2_{L^2(\tilde{\mu}_n)} .
\end{aligned}
$$

Note that the second inequality holds for large T satisfying $\delta = T^{1-2d} < 1/2$. Hence we have

$$
|u(z)|^2 \, e^{-nQ(r_n z)} \le C_2 \, r_n^{-2c} \parallel u \parallel^2_{L^(\tilde{\mu}_n)} ,
$$

where

$$
C_2 \le \delta^{-2} e^{3\alpha\delta^2} \le T^{4d-2} e^{3bT^{-2d}} .
$$

□

Now we consider the case when $c > 0$. Write $c = q + c'$ where q is a non-negative integer and $-1 < c' \le 0$. Then we have the following Lemma.

Lemma 4.2.2. *There exists a constant C_0 such that for all $u \in L^2_a(\tilde{\mu}_n)$ and for large T, we have*

$$
|u(z)|^2 \, e^{-nQ(r_n z)} \le C_0 T^{4d-2c'-2} \, |r_n z|^{-2q} \, r_n^{-2c'} \parallel u \parallel^2_{L^2(\tilde{\mu}_n)} , \quad |z| \le T.
$$

Proof. Replace “$u(z)$” by “$z^q u(z)$” in Lemma 4.2.1. We take $\delta = T^{1-2d}$.

Suppose that $\delta \le |z| \le T$. Repeating the proof in Lemma 4.2.1 for $z^q u(z)$, we obtain the inequality

$$
|z^q u(z)|^2 \; e^{-\tilde{V}_n(z)+\alpha|z|^2} \le 4\delta^{-2} \int_{D(z,\delta/2)} |w^q u(w)|^2 \, e^{-\tilde{V}_n(w)+\alpha|w|^2} \, dA(w),
$$

where α is the number defined in the proof of Lemma 4.2.1. The same argument gives

$$
|u(z)|^2 \, e^{-nQ(r_n z)} \le C_1 r_n^{-2c} \, |z|^{-2q} \parallel u \parallel^2_{L^2(\tilde{\mu}_n)} ,
$$

where

$$
C_1 \le 4\delta^{-2} \, |T + \delta/2|^{2q} \, e^{\alpha\delta T + \alpha\delta^2/4 - 2c\log T} \le C_0 T^{4d-2c'-2}
$$

for some constant C_0 independent of T.

We now suppose that $|z| < \delta$ with $\delta = T^{1-2d} < 1/2$. Repeating the proof of the previous lemma again for $z^q u(z)$, we have

$$|z^q u(z)|^2 e^{-nQ(r_n z)+\alpha|z|^2} \le \delta^{-2} \int_{D(z,\delta)} |w^q u(w)|^2 e^{-nQ(r_n w)+\alpha|w|^2} dA(w).$$

It follows that for $|z| < \delta$,

$$\begin{aligned}|u(z)|^2 e^{-nQ(r_n z)} &\le \delta^{-2} e^{\alpha(2|z|\delta+\delta^2)} r_n^{-2c} |z|^{-2q} (2\delta)^{2q} \| u \|^2_{L^2(\tilde{\mu}_n)} \\ &\le C_2 r_n^{-2c} |z|^{-2q} \| u \|^2_{L^2(\tilde{\mu}_n)},\end{aligned}$$

where $C_2 \le 4^{2q}\delta^{-2+2q} e^{3\alpha\delta^2} \le 4^{2q} T^{(4d-2)(1-q)} e^{3bT^{-2d}}$. □

4.2.2 Local uniform boundedness of the rescaled kernel

Recall that $Q = Q_0 + \operatorname{Re} H + Q_1$ is the canonical decomposition and K_n denote the correlation kernel of the rescaled system $\{z_j\}_1^n$. Write

$$\begin{aligned}K_n(z,w) &= k_n(z,w)\, e^{-\tilde{V}_n(z)/2-\tilde{V}_n(w)/2} \\ &= L_n(z,w)\cdot E_n(z,w)\, |zw|^c,\end{aligned} \tag{4.2.5}$$

where E_n and L_n are defined by

$$\begin{aligned}E_n(z,w) &= e^{-(Q_0(z)+nQ_1(r_n z)+Q_0(w)+nQ_1(r_n w))/2}, \\ L_n(z,w) &= r_n^{2c}\, k_n(z,w)\, e^{-nH(r_n z)/2-n\bar{H}(r_n w)/2}.\end{aligned} \tag{4.2.6}$$

Note that L_n is Hermitian-entire.

Lemma 4.2.3. *Let R_n be the rescaled one-point function $R_n(z) = K_n(z,z)$ and $c = q + c'$ where q is the smallest integer greater than or equal to c. Then the family $\{R_n\}$ has the following upper bound: for large T,*

$$R_n(z) \le C_0 T^{4d-2c'-2} |z|^{2c'}, \quad |z| \le T,$$

where C_0 is the constant which is uniform on n and independent on T.

Proof. Noting that k_n is the reproducing kernel for the space $\mathcal{P}_n(\tilde{\mu}_n)$ where

$d\tilde{\mu}_n = e^{-\tilde{V}_n}\, dA$, we have

$$k_n(z,z) = \sup\{|p(z)|^2 \,:\, p \in \mathcal{P}_n(\tilde{\mu}_n),\, \|\, p\,\|_{L(\tilde{\mu}_n)} \le 1\}.$$

It follows from that Lemma 4.2.2 that for all n and all $|z| \le T$,

$$k_n(z,z) \le C_0 T^{4d-2c'-2}\, |r_n z|^{-2q}\, r_n^{-2c'}\, e^{nQ(r_n z)}.$$

From the definition (4.2.6), we have

$$L_n(z,z) \le C_0 T^{4d-2c'-2}\, |z|^{-2q}\, e^{Q_0(z)+nQ_1(r_n z)}, \quad |z| \le T, \tag{4.2.7}$$

and by (4.2.5),

$$K_n(z,z) \le C_0 T^{4d-2c'-2}\, |z|^{2c'}, \quad |z| \le T.$$

We prove the lemma. $\square$

The following theorem is the main result in this section, which implies Theorem 4.1.2.

Theorem 4.2.4. *There exists a sequence of cocycles c_n such that each subsequence of $\{|z\bar{w}|^{-c'}\,(c_n K_n)(z,w)\}_n$ has a further subsequence converging uniformly on compact subsets of $\mathbb{C}^2$ as $n \to \infty$. Each limit point K of the sequence $\{c_n K_n\}$ has the structure*

$$K(z,w) = L(z,w) e^{-V_0(z)/2 - V_0(w)/2},$$

where $l(z,w) = (z\bar{w})^q L(z,w)$ is Hermitian-entire.

Proof. Write as in (4.2.5)

$$K_n(z,w) = L_n(z,w) \cdot E_n(z,w)\, |zw|^c\,,$$

where E_n and L_n are defined by

$$E_n(z,w) = e^{-(Q_0(z)+nQ_1(r_n z)+Q_0(w)+nQ_1(r_n w))/2},$$
$$L_n(z,w) = r_n^{2c}\, k_n(z,w)\, e^{-nH(r_n z)/2 - n\bar{H}(r_n w)/2}.$$

We take a sequence of cocycles

$$c_n(z,w) = e^{-i\,\mathrm{Im}(H(r_n z)-H(r_n w))/2}$$

and obtain the convergence

$$c_n(z,w)E_n(z,w) = e^{-Q_0(z)/2-Q_0(w)/2}(1+o(1)), \quad n\to\infty, \tag{4.2.8}$$

where $o(1)\to 0$ uniformly on each compact subsets of $\mathbb{C}^2$.

Now we consider the function

$$l_n(z,w) = (z\bar{w})^q L_n(z,w).$$

Then l_n is Hermitian-entire, and by the estimate (4.2.7), we have

$$l_n(z,z) \le C_0 T^{4d-2c'-2} e^{Q_0(z)+nQ_1(r_n z)}, \quad |z|\le T,$$

which implies the family $\{l_n(z,z)\}$ is locally uniformly bounded on $\mathbb{C}$. Thus, by the Cauchy-Schwarz inequality,

$$|l_n(z,w)|^2 \le l_n(z,z) l_n(w,w),$$

the functions $l_n(z,w)$ form a normal family of Hermitian-entire functions. Hence each subsequence of $\{l_n\}$ has a further subsequence which converges locally uniformly to a Hermitian-entire function. Denote by l a limit point of $\{l_n\}$ and write $\lim_{m\to\infty} l_{n_m}\to l$. Put $L(z,w)=(z\bar{w})^{-q}l(z,w)$. By (4.2.8),

$$\begin{aligned} |z\bar{w}|^{-c'}(c_{n_m}K_{n_m})(z,w) &= |z\bar{w}|^q (c_{n_m}L_{n_m}E_{n_m})(z,w) \\ &\to |z\bar{w}|^q\, L(z,w)\, e^{-Q_0(z)/2-Q_0(w)/2} \end{aligned}$$

uniformly on each compact set of $\mathbb{C}^2$. Hence, we have

$$\begin{aligned} K(z,w) &= |z\bar{w}|^c\, L(z,w)\, e^{-Q_0(z)/2-Q_0(w)/2} \\ &= L(z,w)\, e^{-V_0(z)/2-V_0(w)/2}, \end{aligned}$$

which proves the theorem. □

Corollary 4.2.5. *Let $R(z)=K(z,z)$ be a limiting correlation function. Then the convergence $R_{n_m}\to R$ in Theorem 4.2.4 holds in the sense of* *distributions on* $\mathbb{C}$*, and we have the estimate for large* T*,*

$$R(z) \le C_0\, T^{4d-2c'-2}\, |z|^{2c'}\,, \quad |z| \le T. \tag{4.2.9}$$

Proof. By Lemma 4.2.3, we obtain the estimate (4.2.9) and the family $\{R_n\}$ is locally uniformly integrable, which means that for large T, there exists a constant $M = M(T)$ such that for all n

$$\int_{\{|z|\le T\}} R_n(z)\, dA(z) \le M.$$

Thus R_{n_m} converges to R in the sense of distributions on $\mathbb{C}$. □

4.2.3 Positivity

Let L be a limiting kernel in Theorem 4.2.4 and L_0 be the reproducing kernel of $L^2_a(\mu_0)$, where $L^2_a(\mu_0)$ is the subspace of $L^2(\mu_0)$ consisting of all entire functions in $L^2(\mu_0)$ with $d\mu_0(z) = e^{-V_0(z)}\, dA(z)$.

Recall the definition of a positive matrix: a Hermitian function F is called a positive matrix if $\sum_{i,j=1}^{N} \alpha_i \bar{\alpha}_j F(z_i, z_j) \le 0$ for all points $z_j \in \mathbb{C}$ and all scalars $\alpha_j \in \mathbb{C}$.

Now we prove the "positivity" theorem. (cf. Section 3.4.2.)

Theorem 4.2.6. *L is a reproducing kernel for the Hilbert space $\mathcal{H}_L$, which is the completion of the linear span of $\{L_z\}_{z\in\mathbb{C}}$ equipped with the inner product $\langle L_z, L_w\rangle_{\mathcal{H}_L} = L(w,z)$. Moreover, $L_0 - L$ is a positive matrix.*

Proof. First, we consider the function L_n defined in (4.2.6). Then L_n is the reproducing kernel for the Hilbert space $\mathcal{H}_n$ of entire functions

$$\mathcal{H}_n = \{f \,:\, f(z) = p(z)\cdot e^{-nH(r_n z)/2},\, p \in \mathrm{Pol}(n)\}$$

with the norm of $L^2(\mu_{0,n})$. Here,

$$d\mu_{0,n}(z) = e^{-nQ_0(r_n z) - nQ_1(r_n z) + 2c\log|z|}\, dA(z)$$

and $\mathrm{Pol}(n)$ is the linear space of holomorphic polynomials of degree $\le n-1$. Indeed, for $f(z) = p(z)\cdot e^{-nH(r_n z)/2} \in \mathcal{H}_n$,

$$\int f(z)\overline{L_n(z,w)} d\mu_{0,n}(z) = e^{-nH(r_n w)/2} \int p(z)\overline{k_n(z,w)} e^{-\tilde{V}_n(z)} dA(z) = f(w).$$

Each L_n is a positive matrix since L_n is the reproducing kernel for $\mathcal{H}_n$. Thus, a limit point L is also a positive matrix. By Theorem 3.4.2, L is the reproducing kernel for the Hilbert space $\mathcal{H}_L$.

To show that $L_0 - L$ is a positive matrix, consider the norm of functions L_z ($z \in \mathbb{C}$) which generate $\mathcal{H}_L$. Fatou's Lemma gives

$$\begin{aligned}
\| L_z \|_{\mathcal{H}_L}^2 &= L(z,z) = \lim_{m\to\infty} L_{n_m}(z,z) \\
&= \lim_{m\to\infty} \int |L_{n_m}(w,z)|^2 \, e^{-Q_0(w)-nQ_1(r_n w)+2c\log|w|} dA(w) \\
&\geq \int |L(w,z)|^2 \, e^{-V_0(w)} \, dA(w) = \| L_z \|_{L^2(\mu_0)}^2 ,
\end{aligned}$$

whence $\mathcal{H}_L$ is contractively embedded in $L_a^2(\mu_0)$. Thus, by Aronszajn's theorem on differences of reproducing kernels in [8], $L_0 - L$ is a positive matrix. □

4.3 Homogeneous singularities

In this section, we prove Theorem 4.1.4. We assume that Q is a homogeneous polynomial of degree $2d$. Then, by the canonical decomposition, we have $Q(\zeta) = Q_0(\zeta) + \alpha \operatorname{Re} \zeta^{2d}$ for $\alpha \in \mathbb{C}$. Here, $H(\zeta) = \alpha\, \zeta^{2d}$ and $Q_1 = 0$ in the canonical decomposition of Q.

In this case, the holomorphic kernel L_n in (4.2.6) is the reproducing kernel for the space

$$\mathcal{H}_n = \{ f \,:\, f(z) = p(z) \cdot e^{-\alpha\, z^{2d}/2},\, p \in \operatorname{Pol}(n) \}$$

equipped with the norm of $L^2(\mu_0)$ where

$$d\mu_0(z) = e^{-V_0(z)} dA(z) = e^{-Q_0(z)+2c\log|z|} dA(z).$$

Since the inclusion $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ is isometric, every limit point must be the same for all subsequences. Thus, there exists a unique limiting kernel L such that $\lim L_n = L$.

We note that for all polynomial p and $z \neq 0$,

$$p(z) = \lim_{n\to\infty} \int p(w) L_n(z,w)\, d\mu_0(w) = \int p(w) L(z,w)\, d\mu_0(w).$$

Since polynomials are dense in $L^2_a(\mu_0)$, we have for all $f \in L^2_a(\mu_0)$ and $z \neq 0$,

$$f(z) = \lim_{m\to\infty} \int p_m(w)L(z,w)\,d\mu_0(w) = \int f(w)L(z,w)\,d\mu_0(w),$$

where $\{p_m\}$ is a sequence of polynomials such that $p_m \to f$ as $m \to \infty$. Hence $L(z,w) = L_0(z,w)$ for z, $w \in \mathbb{C} \setminus \{0\}$, where L_0 is the reproducing kernel for $L^2_a(\mu_0)$.

4.4 Johansson's marginal measure theorem

In this section, we prove the Proposition 4.1.1, using an argument inspired by [25] and [27]. Recall the partition function Z_n of the system is given by

$$Z_n = \int_{\mathbb{C}} e^{-H_n(\mathbf{z})} dA^{\otimes n}(\mathbf{z}), \quad \mathbf{z} \in (z_j) \in \mathbb{C}^n,$$

where

$$H_n(\mathbf{z}) = -\sum_{j\neq k} \log|z_j - z_k| + \sum_{j=1}^{n} nV_n(z_j).$$

Let σ_Q be the equilibrium measure associated with the potential Q and $I_Q[\sigma_Q]$ be the logarithmic energy of σ_Q, see Section 2.3.1. By the definition of σ_Q,

$$I_Q(\sigma_Q) = \inf\{I_Q(\mu)\,|\,\mu \in \mathcal{P}_c(\mathbb{C})\}, \tag{4.4.1}$$

where $\mathcal{P}_c(\mathbb{C})$ is the collection of all positive, compactly supported Borel probability measures on $\mathbb{C}$. We first prove the following lemma.

Lemma 4.4.1. *We have*

$$\frac{1}{n(n-1)} \log Z_n \to -\frac{1}{2}\, I_Q[\sigma_Q] \quad \text{as} \quad n \to \infty. \tag{4.4.2}$$

In order to prove Lemma 4.4.1, we need some preliminary definitions. We write

$$L_Q(z,w) = \log \frac{1}{|z-w|^2} + Q(z) + Q(w)$$

and observe that the weighted logarithmic energy of μ can be written as

$$I_Q[\mu] = \iint_{\mathbb{C}^2} L_Q(z,w)\, d\mu(z) d\mu(w).$$

We now set

$$L_n^Q := \inf_{z_1,\cdots,z_n\in\mathbb{C}} \frac{1}{n(n-1)} \sum_{1\le j\ne k\le n} L_Q(z_j,z_k), \tag{4.4.3}$$

and the infimum is attained at some set $\mathcal{F}_n = \{z_1,\cdots,z_n\}$. We call $\mathcal{F}_n$ n-th weighted Fekete sets with respect to Q. Then $\{L_n^Q\}$ is an increasing sequence and

$$L_n^Q \to I_Q[\sigma_Q] \quad \text{as} \quad n\to\infty. \tag{4.4.4}$$

See [34], Chapter 3.

Proof of Lemma 4.4.1. We first show that

$$\liminf_{n\to\infty} \frac{1}{n(n-1)} \log Z_n \ge -\frac{1}{2} I_Q[\sigma_Q]. \tag{4.4.5}$$

Write Z_n in the form

$$\begin{aligned} Z_n &= \int_{\mathbb{C}^n} e^{-\frac{1}{2}\sum_{j\ne k}\log\frac{1}{|z_j-z_k|^2}-\sum_{j=1}^n nV_n(z_j)} dA^{\otimes n}(\mathbf{z}) \\ &= \int_{\mathbb{C}^n} e^{-\frac{1}{2}\sum_{j\ne k} L_Q(z_j,z_k)-\sum_{j=1}^n (Q(z_j)-2c\log|z_j-p|)} dA^{\otimes n}(\mathbf{z}). \end{aligned}$$

Let $V_0(z) = Q(z) - 2c\log|z-p|$. By inserting the density $\rho_Q = \frac{1}{\pi}\chi_{S_Q}\Delta Q$ of the equilibrium measure σ_Q, we obtain

$$Z_n \ge \int_{\mathbb{C}^n} e^{-\frac{1}{2}\sum_{j\ne k} L_Q(z_j,z_k)-\sum_{j=1}^n V_0(z_j)-\sum_{j=1}^n \log\rho_Q(z_j)} \prod_{k=1}^n d\sigma_Q(z_k).$$

Using the Zensen's inequality, we have the following inequality:

$$\begin{aligned} &\log Z_n \\ &\ge \int_{\mathbb{C}^n} \Big(-\frac{1}{2}\sum_{j\ne k} L_Q(z_j,z_k) - \sum_{j=1}^n V_0(z_j) - \sum_{j=1}^n \log\rho_Q(z_j)\Big) \prod_{k=1}^n d\sigma_Q(z_k) \\ &\ge -\frac{1}{2}\cdot n(n-1) \iint_{\mathbb{C}^2} L_Q(z,w)\, d\sigma_Q(z) d\sigma_Q(w) \\ &\quad - n\int_{\mathbb{C}} (Q(z) - 2c\log|z-p|)\, d\sigma_Q(z) - n\int_{\mathbb{C}} \log\Delta Q(z) d\sigma_Q(z). \end{aligned}$$

It proves (4.4.5) since

$$I_Q[\sigma_Q] = \iint_{\mathbb{C}^2} L_Q(z,w)\, d\sigma_Q(z) d\sigma_Q(w).$$

To prove the opposite direction, we write

$$\begin{aligned} Z_n &= \int_{\mathbb{C}^n} e^{-\frac{1}{2}\sum_{j\neq k} L_Q(z_j,z_k)} e^{-\sum_{j=1}^n (Q(z_j)-2c\log|z_j-p|)} d\,\mathrm{Vol}_n(z) \\ &\le e^{-\frac{n}{2}(n-1)L_n^Q} \left(\int_{\mathbb{C}} e^{-(Q(z)-2c\log|z-p|)} dA(z) \right)^n, \end{aligned}$$

where L_n^Q is the infimum defined in (4.4.3). Here, for each $c > -1$

$$\int_{\mathbb{C}} e^{-(Q(z)-2c\log|z-p|)} dA(z)$$

is bounded by some constant M_Q. Since the sequence $\{L_n^Q\}$ converges to $I_Q[\sigma_Q]$ as $n \to \infty$, we obtain

$$\frac{1}{n(n-1)} \log Z_n \le -\frac{1}{2} L_n^Q + \frac{1}{n-1} \log M_Q,$$

whence

$$\limsup_{n\to\infty} \frac{1}{n(n-1)} \log Z_n \le -\frac{1}{2} I_Q[\sigma_Q].$$

$\square$

For given $\epsilon > 0$, We define a set $\mathcal{A}_n(\epsilon)$ by

$$\mathcal{A}_n(\epsilon) = \left\{ \mathbf{z} = (z_j) \in \mathbb{C}^n \ : \ \frac{1}{n(n-1)} \sum_{j\neq k} L_Q(z_j,z_k) \le I_Q[\sigma_Q] + \epsilon \right\}. \tag{4.4.6}$$

From this setting, we have the following lemma.

Lemma 4.4.2. *As $n \to \infty$,*

$$\mathbf{P}_n(\mathcal{A}_n(\epsilon)) \ge 1 - e^{-\frac{1}{2}n(n-1)\epsilon + O(n)}.$$

Proof. We write

$$\begin{aligned}&\mathbf{P}_n\left(\mathbb{C}^n \setminus \mathcal{A}_n(\epsilon)\right)\\ &= \frac{1}{Z_n}\int_{\mathbb{C}^n\setminus\mathcal{A}_n(\epsilon)} e^{-\frac{1}{2}\sum_{j\neq k} L_Q(z_j,z_k)-\sum_{j=1}^n V_0(z_j)} dA^{\otimes n}(\mathbf{z}).\end{aligned}$$

Taking the logarithm to $\mathbf{P}_n(\mathbb{C}^n \setminus \mathcal{A}_n(\epsilon))$, we obtain

$$\begin{aligned}&\log \mathbf{P}_n\left(\mathbb{C}^n \setminus \mathcal{A}_n(\epsilon)\right)\\ =&\log \int_{\mathbb{C}^n\setminus\mathcal{A}_n(\epsilon)} e^{-\frac{1}{2}\sum_{j\neq k} L_Q(z_j,z_k)-\sum_{j=1}^n V_0(z_j)} dA^{\otimes n}(\mathbf{z}) - \log Z_n\\ \leq& -\frac{n}{2}(n-1)\left(I_Q[\sigma_Q]+\epsilon\right) - \log Z_n + \log\int_{\mathbb{C}^n\setminus\mathcal{A}_n(\epsilon)} e^{-\sum_{j=1}^n V_0(z_j)} dA^{\otimes n}(\mathbf{z}).\end{aligned}$$

It is shown in the proof of Lemma 4.4.1 that the asymptotic

$$\log Z_n = -\frac{n}{2}(n-1)I_Q[\sigma_Q] + O(n), \quad n\to\infty, \tag{4.4.7}$$

holds. The integrability of the function $e^{-V_0(z)}$ gives the upper bound

$$\int_{\mathbb{C}^n\setminus\mathcal{A}_n(\epsilon)} e^{-\sum_{j=1}^n V_0(z_j)} dA^{\otimes n}(\mathbf{z}) \leq \left(M_Q\right)^n, \tag{4.4.8}$$

where M_Q is some positive constant depending on Q. The asymptotic of $\log Z_n$ (4.4.7) and the upper bound (4.4.8) show that

$$\log \mathbf{P}_n(\mathbb{C}\setminus\mathcal{A}_n(\epsilon)) \leq -\frac{n}{2}(n-1)\epsilon + O(n)$$

holds asymptotically as $n\to\infty$. □

Lemma 4.4.3 (Proposition 3.2, [25]). *For a positive real number R and $\mathbf{z}\in\mathcal{A}_n(\epsilon)$, let*

$$n_R(\mathbf{z}) = \#\{j\in\{1,\cdots,n\} : \mathbf{z}=(z_j)_1^n,\ |z_j|\leq R\}.$$

Then for any $0<\epsilon'<1$, there exists sufficiently large R such that

$$\frac{n_R(\mathbf{z})}{n} > 1-\epsilon'.$$

Proof. Split the sum as follows:

$$\sum_{j\neq k} L_Q(z_j,z_k) = \sum_{\substack{j\neq k,\\ |z_j|,|z_k|\le R}} L_Q(z_j,z_k) + 2\sum_{\substack{j\neq k,\\ |z_j|\le R,\\ |z_k|>R}} L_Q(z_j,z_k) + \sum_{\substack{j\neq k,\\ |z_j|,|z_k|>R}} L_Q(z_j,z_k).$$

For simplicity, write $n_R(\mathbf{z}) = n_R$. The first term on the right-hand side has the following lower bound:

$$\sum_{\substack{j\neq k,\\ |z_j|,|z_k|\le R}} L_Q(z_j,z_k) \ge n_R(n_R-1)\, I_Q[\sigma_Q]. \tag{4.4.9}$$

The growth condition (2.1.2) implies that there exist δ, $C>0$ such that for sufficiently large R,

$$\begin{aligned} Q(z) &\ge (2+\delta)\log|z| - C, \quad \text{if} \quad |z|>R,\\ Q(z) &\ge -C, \quad \text{if} \quad z\in\mathbb{C}. \end{aligned}$$

We also choose sufficiently large R such that for $|z_j|\le R$ and $|z_k|>R$,

$$\begin{aligned} -\log|z_j-z_k|^2 + Q(z_j) + Q(z_k) &\ge -\log\left|\frac{z_k-z_j}{z_k}\right|^2 + \delta\log R - 2C\\ &\ge I_Q[\sigma_Q] + \frac{\delta}{2}\log R \end{aligned}$$

and for $R<|z_j|<|z_k|$,

$$\begin{aligned} -\log|z_j-z_k|^2 + Q(z_j) + Q(z_k) &\ge -\log\left|\frac{z_k-z_j}{z_k}\right|^2 + (2+2\delta)\log R - 2C\\ &\ge I_Q[\sigma_Q] + \delta\log R. \end{aligned}$$

Thus for sufficiently large R, we have

$$\begin{aligned} \sum_{\substack{j\neq k,\\ |z_j|\le R,\, |z_k|>R}} L_Q(z_j,z_k) &= \sum_{\substack{j\neq k,\\ |z_j|\le R,\, |z_k|>R}} \Big(-\log|z_j-z_k|^2 + Q(z_j)+Q(z_k)\Big) \qquad (4.4.10)\\ &\ge n_R\,(n-n_R)\cdot\left(I_Q[\sigma_Q] + \frac{\delta}{2}\log R\right) \end{aligned}$$

and

$$\sum_{\substack{j\neq k,\\ |z_j|,|z_k|>R}} L_Q(z_j,z_k) \geq (n-n_R)(n-n_R-1)\cdot(I_Q[\sigma_Q]+\delta\log R). \qquad (4.4.11)$$

By (4.4.9), (4.4.10), and (4.4.11), we obtain the following inequality:

$$n(n-1)I_Q[\sigma_Q]+(n-n_R)(n-1)\log R \leq \sum_{j\neq k} L_Q(z_j,z_k).$$

Since $\mathbf{z}=(z_j)_1^n \in \mathcal{A}_n(\epsilon)$, we have

$$\frac{n-n_R}{n}\log R \leq \epsilon,$$

which proves the lemma. □

Proof of Theorem 4.1.1. Let f be a continuous and bounded function defined in $\mathbb{C}$. Then, for given $\epsilon>0$, we obtain

$$\begin{aligned}\int_{\mathbb{C}} f\ d\sigma_n &= \frac{1}{n}\int_{\mathbb{C}^n}\sum_{j=1}^n f(z_j)\ d\mathbf{P}_n(\mathbf{z})\\ &= \frac{1}{n}\int_{\mathbb{C}^n\setminus\mathcal{A}_n(\epsilon)}\sum_{j=1}^n f(z_j)\ d\mathbf{P}_n(\mathbf{z}) + \frac{1}{n}\int_{\mathcal{A}_n(\epsilon)}\sum_{j=1}^n f(z_j)\ d\mathbf{P}_n(\mathbf{z}),\end{aligned}$$

where $\mathbf{z}=(z_1,\cdots,z_n)\in\mathbb{C}^n$. Since $\mathbf{P}_n(\mathbb{C}^n\setminus\mathcal{A}_n(\epsilon))$ decays exponentially, the first integral becomes negligible as $n\to\infty$.

On the other hand, we note that for each $\mathbf{z}=(z_1,\cdots,z_n)\in\mathcal{A}_n(\epsilon)$, the following inequalities holds:

$$I_Q[\sigma_Q]-\epsilon_n \leq \frac{1}{n(n-1)}\sum_{j\neq k} L_Q(z_j,z_k) \leq I_Q[\sigma_Q]+\epsilon, \qquad (4.4.12)$$

where ϵ_n is independent on z and $\epsilon_n\to 0$ as $n\to\infty$. Indeed, the well-known convergence (4.4.4) implies the first inequality and the second inequality is obtained directly from the definition of the set $\mathcal{A}_n(\epsilon)$.

We now let ϵ in (4.4.12) tend to 0 with the rate $O(n^{-1/2})$ as $n\to\infty$ and fix a compact subset $K\subset\mathbb{C}$ such that it contains S_Q and $(n-n_K)/n$ is sufficiently small where n_K is the number of z_j which is contained in K for $\mathbf{z}=(z_j)_1^n\in\mathcal{A}_n(\epsilon)$.

For $\mathbf{z} = (z_j)_1^n \in \mathcal{A}_n(\epsilon)$, consider the sequence of measures

$$\nu_n = \frac{1}{n} \sum_{z_j \in K} \delta_{z_j}.$$

Then any subsequence of ν_n has a weak-star convergent subsequence ν_{n_k} converging to some $\nu^* \in \mathcal{P}_c(\mathbb{C})$. We claim that $I_Q[\nu^*] = I_Q[\sigma_Q]$.

For a positive number M, we set

$$l_M(z, w) = \min\{M, L_Q(z, w)\}.$$

Note that l_M is bounded and continuous. We have

$$\begin{aligned} I_Q[\nu^*] &= \iint L_Q(z, w)\, d\nu^*(z)\, d\nu^*(w) \\ &= \lim_{M\to\infty} \iint l_M(z, w)\, d\nu^*(z)\, d\nu^*(w) \\ &= \lim_{M\to\infty} \lim_{k\to\infty} \iint l_M(z, w)\, d\nu_{n_k}(z)\, d\nu_{n_k}(w). \end{aligned}$$

Since for a positive integer m,

$$\begin{aligned} \iint l_M(z, w)\, d\nu_m(z)\, d\nu_m(w) &= \frac{1}{m^2} \sum_{j=1}^{m} l_M(z_j, z_j) + \frac{1}{m^2} \sum_{j\neq k} l_M(z_j, z_k) \\ &\leq \frac{M}{m} + \frac{1}{m^2} \sum_{j\neq k} L_Q(z_j, z_k). \end{aligned}$$

It follows from (4.4.12) that $I_Q[\nu^*] \leq I_Q[\sigma_Q]$. Clearly $I_Q[\nu^*] \geq I_Q[\sigma_Q]$ by (4.4.1), and hence $I_Q[\nu^*] = I_Q[\sigma_Q]$. The unicity of the equilibrium measure implies that $\nu^* = \sigma_Q$.

Thus we obtain the convergence for $\mathbf{z} = (z_j)_1^n \in \mathcal{A}_n(\epsilon)$,

$$\frac{1}{n} \sum_{j=1}^{n} f(z_j) \to \int_{\mathbb{C}} f\, d\sigma_Q, \quad (n \to \infty).$$

Since $\mathbf{P}_{n,\beta}^{(c)}(\mathcal{A}_n(\epsilon)) \geq 1 - e^{-\frac{\beta}{2} n(n-1)\epsilon + O(n)}$ as $n \to \infty$, we obtain the convergence

$$\frac{1}{n}\int_{\mathcal{A}_n(\epsilon)}\sum_{j=1}^{n} f(z_j)\ d\mathbf{P}_{n,\beta}^{(c)}(\mathbf{z}) \to \int_{\mathbb{C}} f\ d\sigma_Q, \quad n \to \infty.$$

Hence we prove the theorem. $\square$

Chapter 5

Asymptotics for the one point functions

This chapter is devoted to prove some asymptotics for limiting one point functions $R(z) = K(z,z)$. More precisely, using the modified Hörmander estimate, we prove the asymptotic growth of $R(z)$ when z tends to ∞. This chapter is based on [6, 7].

5.1 Bulk singularities

In this section, we consider the system of eigenvalues $\{\zeta_j\}_{j=1}^n$ associated with the potential which has a bulk singularity of the type $2d-2$ at 0. Recall the canonical decomposition of $Q = Q_0 + \operatorname{Re} H + Q_1$ and the microscopic scale r_n defined in Section 3. Let L be a limiting holomorphic kernel in Theorem 3.2.1 and L_0 be the reproducing kernel for the space $L^2_a(\mu_0)$ where $d\mu_0(z) = e^{-Q_0(\tau_0 z)}dA(z)$ in Section 3.

We prove the following estimates.

Theorem 5.1.1. *Let $R(z) = L(z,z)\,e^{-Q_0(\tau_0 z)}$ be a limiting one point function and $R_0(z) = L_0(z,z)\,e^{-Q_0(\tau_0 z)}$. Then, we have as $z \to \infty$,*

(i) $R_0(z) = \Delta_z[Q_0(\tau_0 z)] \cdot (1 + O(z^{1-d}))$,

(ii) $R(z) \ = \Delta_z[Q_0(\tau_0 z)] \cdot (1 + O(z^{1-d}))$.

Part (i) depends on an estimate of the Bergman kernel for the space of $L^2_a(\mu_0)$, and the method of "approximate Bergman projection" is used for part (ii). From now on we assume that $\tau_0 = 1$ for simplicity.

5.1.1 Asymptotics for $L_0(z,z)$

In this section, we prove part (i) of Theorem 5.1.1.

Let $A_0(z,w)$ be the Hermitian polynomial such that $A_0(z,z)=Q_0(z)$ and put

$$L_0^\sharp(z,w)=\left[\partial_1\bar\partial_2 A_0\right](z,w)\cdot e^{A_0(z,w)}.$$

We write $L_z^\sharp(w)$ for $L_0^\sharp(w,z)$ and, for suitable functions u,

$$\pi^\sharp u(z)=\langle u,L_z^\sharp\rangle_{L^2(\mu_0)}=\int_{\mathbb{C}} u\bar L_z^\sharp e^{-Q_0}\,dA.$$

We fix a point $z\in\mathbb{C}$ with $|z|$ large enough and a number $\delta_0=\delta_0(z)>0$. We here write χ_z for a fixed C^∞-smooth function with $\chi_z(w)=1$ when $|w-z|\le\delta_0$ and $\chi_z(w)=0$ when $|w-z|\ge 2\delta_0$.

We first prove the following estimate.

Lemma 5.1.2. *If $|1-w/z|$ is sufficiently small, then*

$$2\operatorname{Re}A_0(z,w)\le Q_0(z)+Q_0(w)-c|z|^{2d-2}|w-z|^2$$

where c is a positive constant.

Proof. Put $h=w-z$. By Taylor series expansion, we have

$$A_0(w,z)=Q_0(z)+\sum_1^{2d}\frac{\partial^j Q_0(z)}{j!}h^j.$$

Similarly, $A_0(w,w)=Q_0(z)+\sum_{i+j\ge1}\frac{\partial^i\bar\partial^j Q_0(z)}{i!j!}h^i\bar h^j$. Hence

$$2\operatorname{Re}A_0(z,w)-Q_0(z)-Q_0(w)+\Delta Q_0(z)|h|^2=-\sum_{i,j\ge1,i+j\ge3}\frac{\partial^i\bar\partial^j Q_0(z)}{i!j!}h^i\bar h^j. \tag{5.1.1}$$

Since Q_0 is homogeneous of degree $2d$, the derivative $\partial^i\bar\partial^j Q_0$ is homogeneous of degree $2d-i-j$. Hence

$$\left|\partial^i\bar\partial^j Q_0(z)\right||w-z|^{i+j}\le C|z|^{2d-2}|w-z|^2|1-w/z|^{i+j-2}.$$

Thus, if $i+j\ge3$ and $|1-w/z|$ is sufficiently small, then the left hand side in (5.1.1) is dominated by an arbitrarily small multiple of $|z|^{2d-2}|z-w|^2$. On the other hand, by homogeneity and positive definiteness of ΔQ_0, we have that $\Delta Q_0(z)|z-w|^2 \geq c'|z|^{2d-2}|z-w|^2$ where c' is a positive constant. The lemma thus follows with any positive constant $c < c'$. $\square$

As always, we write $d\mu_0 = e^{-Q_0}\, dA$. $L^2_a(\mu_0)$ denotes the associated Bergman space of entire functions and L_0 is the Bergman kernel of that space.

Lemma 5.1.3. *Suppose that $|z| \geq 1$ and let δ_0 be a positive number with $\delta_0/|z|$ sufficiently small. Then there is a constant $C = C(\delta_0)$ such that, for all functions $u \in L^2_a(\mu_0)$*

$$\left| u(z) - \pi^\sharp[\chi_z u](z) \right| \leq C\|\, u\,\|_{L^2(\mu_0)}\,(\delta_0^{-1}+1)e^{\,Q_0(z)/2}.$$

Proof. Note that

$$\begin{aligned}
&\pi^\sharp[\chi_z u](z)\\
&= \int_{\mathbb{C}} \chi_z(w)u(w)\left[\partial_1\bar{\partial}_2 A_0\right](z,w)\cdot e^{\,A_0(z,w)-A_0(w,w)}\, dA(w)\\
&= -\int_{\mathbb{C}} \frac{u(w)\chi_z(w)F(z,w)}{w-z}\bar{\partial}_w\left[e^{\,A_0(z,w)-A_0(w,w)}\right] dA(w),
\end{aligned} \tag{5.1.2}$$

where

$$F(z,w) = \frac{(w-z)\left[\partial_1\bar{\partial}_2 A_0\right](z,w)}{\bar{\partial}_2 A_0(w,w) - \bar{\partial}_2 A_0(z,w)}. \tag{5.1.3}$$

Now fix w. By Taylor's formula, the denominator $P(z) = \bar{\partial}_2 A_0(w,w) - \bar{\partial}_2 A_0(z,w)$ is equal to the polynomial

$$-\Delta Q_0(w)\cdot(z-w) - \frac{\partial\Delta Q_0(w)}{2}\cdot(z-w)^{\,2} - \cdots - \frac{\partial^{d-1}\Delta Q_0(w)}{d!}\cdot(z-w)^{\,d}.$$

Here the derivative $\partial^j\Delta Q_0(w) = |w|^{\,2d-2-j}\,\partial^j\Delta Q_0(w/|\,w\,|)$ is positively homogeneous of degree $2d-2-j$. Put $c(w) = \Delta Q_0(w/|\,w\,|)$. We then have that

$$P(z) = c(w)|\,w\,|^{\,2d-2}\cdot(w-z) + O(\,(w-z)^{\,2}\,) \quad \text{as} \quad z \to w.$$

Since also $\partial_1\bar{\partial}_2 A_0(z,w) = c(z)\,|z|^{\,2d-2}\,(1+O(w-z))$, we have by (5.1.3)

$$F(z,w) = 1 + O(w-z) \quad \text{as} \quad w \to z. \tag{5.1.4}$$

By the form of F it is also clear that

$$\bar{\partial}_2 F(z,w) = O(w-z) \quad \text{as} \quad w \to z. \tag{5.1.5}$$

An integration by parts in (5.1.2) gives $\pi^\sharp[\chi_z u](z) = u(z) + \epsilon_1 + \epsilon_2$ where

$$\begin{aligned}
\epsilon_1 &= \int \frac{u(w)\bar{\partial}\chi_z(w)F(z,w)}{w-z}\, e^{A_0(z,w)-A_0(w,w)}\, dA(w), \\
\epsilon_2 &= \int \frac{u(w)\chi_z(w)\bar{\partial}_2 F(z,w)}{w-z}\, e^{A_0(z,w)-A_0(w,w)}\, dA(w).
\end{aligned}$$

Inserting the estimates (5.1.4) and (5.1.5), using also that $\bar{\partial}\chi_z(w) = 0$ when $|\,w-z\,| \le \delta_0$ we find that

$$\begin{aligned}
|\epsilon_1| &\le C\delta_0^{-1} \int |u(w)| \left|\bar{\partial}\chi_z(w)\right| e^{\operatorname{Re} A_0(z,w) - Q_0(w)}\, dA(w), \\
|\epsilon_2| &\le C \int \chi_z(w)\, |u(w)|\, e^{\operatorname{Re} A_0(z,w) - Q_0(w)}\, dA(w).
\end{aligned}$$

To estimate ϵ_1 we use Lemma 5.1.2 to get

$$e^{\operatorname{Re} A_0(z,w) - Q_0(w)/2} \le C e^{Q_0(z)/2 - c|z-w|^2}. \tag{5.1.6}$$

This gives

$$\begin{aligned}
|\epsilon_1|\, e^{-Q_0(z)/2} &\le C\delta_0^{-1} \int |u(w)|\,|\,\bar{\partial}\chi_z(w)\,| e^{-Q_0(w)/2}\, dA(w) \\
&\le C\delta_0^{-1} \|\, u\,\|_{L^2(\mu_0)} \|\, \bar{\partial}\chi_z\,\|_{L^2} \le C' \|\, u\,\|_{L^2(\mu_0)}.
\end{aligned}$$

To estimate ϵ_2 we note that by (5.1.6)

$$|\epsilon_2|\, e^{-Q_0(z)/2} \le C\|\, u\,\|_{L^2(\mu_0)} \left(\int_{|\,w-z\,| \le 2\delta_0} e^{-c|z-w|^2} dA(w) \right)^{1/2} \le C\|\, u\,\|_{L^2(\mu_0)}.$$

The proof is complete. $\square$

Let $\pi_0 : L^2(\mu_0) \to L^2_a(\mu_0)$ be the Bergman projection, $\pi_0[f](z) = \langle f, L_z \rangle_{L^2(\mu_0)}$, where we write $L_z(w)$ for $L_0(w,z)$. Noting that

$$(\pi^\sharp[\chi_z L_z](z))^* = \langle \chi_z L_z, L_z^\sharp \rangle^* = \langle \chi_z L_z^\sharp, L_z \rangle = \pi_0[\chi_z L_z^\sharp](z),$$

we see that

$$\left|L_z(z) - \pi_0[\chi_z L_z^\sharp](z)\right| = \left|L_z(z) - \pi^\sharp[\chi_z L_z](z)\right|.$$

If we choose $u = L_z$ in Lemma 5.1.3 and recall that $\| L_z \|^2_{L^2(\mu_0)} = L_0(z, z)$, we obtain the estimate

$$\left|L_0(z, z) - \pi_0[\chi_z L_z^\sharp](z)\right| \le C\sqrt{L_0(z, z)} \cdot e^{Q_0(z)/2}, \qquad |z| \ge 1. \tag{5.1.7}$$

Lemma 5.1.4. *There is a constant C such that for all $|z| \ge 1$ and all $\delta_0 = \delta_0(z) > 0$ with $\delta_0/|z|$ small enough*

$$\left|\Delta Q_0(z)\, e^{Q_0(z)} - \pi_0 \left[\chi_z L_z^\sharp\right](z)\right| \le C|\, z\,|^{d-1} e^{Q_0(z)}.$$

Proof. Consider the function $u_0 = \chi_z L_z^\sharp - \pi_0[\chi_z L_z^\sharp]$. This is the norm-minimal solution in $L^2(\mu_0)$ to the problem $\bar\partial u = (\bar\partial \chi_z) \cdot L_z^\sharp$.

Since Q_0 is strictly subharmonic on the support of χ_z we can apply the standard Hörmander estimate to obtain (see [26])

$$\begin{aligned}
\|u\|^2_{L^2(\mu_0)} &\le \int_{\mathbb{C}} \left|\bar\partial \chi_z\right|^2 \left|L_z^\sharp\right|^2 \frac{e^{-Q_0}}{\Delta Q_0}\, dA \\
&\le C\, |z|^{-(2d-2)} \|\bar\partial \chi_z\|^2_{L^2} \sup_{\delta_0 \le |w-z| \le 2\delta_0} \left|[\partial_1 \bar\partial_2 A_0](z, w)\right|^2 e^{2\,\mathrm{Re}\, A_0(z,w) - A_0(w,w)},
\end{aligned}$$

where we use the homogeneity of ΔQ_0 in the second inequality.

By Taylor's formula and (5.1.6), we have when $\delta_0 \le |w - z| \le 2\delta_0$

$$\left|[\partial_1 \bar\partial_2 A_0](z, w)\right|^2 e^{2\,\mathrm{Re}\, A_0(z,w) - A_0(w,w)} \le C \Delta Q_0(z)^2 e^{Q_0(z) - 2c|z|^{2d-2}|z-w|^2}.$$

By the homogeneity of ΔQ_0 we thus obtain the estimate

$$\|\, u \,\|_{L^2(\mu_0)} \le C|\, z\,|^{d-1} e^{Q_0(z)/2 - c'\delta_0^2 |z|^{2d-2}}. \tag{5.1.8}$$

We now pick another number $\delta > 0$ and invoke the following pointwise-L^2 estimate (see [5, Lemma 3.1])

$$|u(z)|^2\, e^{-Q_0(z)} \le C e^{c''\delta \Delta Q_0(z)|z|} \delta^{-2} \int_{D(z,\delta)} |u(w)|^2\, e^{-Q_0(w)}\, dA(w). \tag{5.1.9}$$

Combining with (5.1.8), this gives

$$|u(z)|^2 e^{-Q_0(z)} \le C\delta^{-2} e^{-c'\delta_0^2|z|^{2d-2}+c''\delta|z|^{2d-1}} |z|^{2d-2} e^{Q_0(z)}.$$

Choosing δ_0 a small multiple of $|z|^{1/2}$ and then δ small enough, we insure that the right hand side is dominated by $C|z|^{2d-2}e^{Q_0(z)}$, as desired. □

Proof of Part (i) of Theorem 5.1.1. By the estimate (5.1.7) and Lemma 5.1.4 we have

$$\left|\Delta Q_0(z)\, e^{Q_0(z)} - L_0(z,z)\right| \le C_1\sqrt{L_0(z,z)}e^{Q_0(z)/2} + C_2|z|^{d-1}e^{Q_0(z)}.$$

Writing $R_0(z) = L_0(z,z)e^{-Q_0(z)}$, this becomes

$$\left||z|^{2d-2}c(z) - R_0(z)\right| \le C_1\sqrt{R_0(z)} + C_2|z|^{d-1}, \tag{5.1.10}$$

where $c(z) = \Delta Q_0(z/|z|)$. We must prove that the left hand side in (5.1.10) is dominated by $M\,|z|^{1-d}\,\Delta Q_0(z)$ for all large $|z|$, where M is a suitable constant. If this is false, there are two possibilities. First we assume that $R_0(z) \le (1 - M|z|^{1-d})\Delta Q_0(z)$ for arbitrarily large $|z|$. By (5.1.10),

$$M\,|z|^{d-1}\,c(z) \le C_1\sqrt{R_0(z)} + C_2|z|^{d-1} \le (C_1' + C_2)|z|^{d-1},$$

and we reach a contradiction for large enough M.

In the remaining case we have $R_0(z) \ge (1 + M|\,z\,|^{1-d})\Delta Q_0(z)$. Then (5.1.10) gives the estimate $R_0(z) \ge cM^2|\,z\,|^{2d-2}$ for some $c > 0$. Since $\Delta Q_0(z) \le c'|\,z\,|^{2d-2}$ for some $c' > 0$, we obtain

$$R_0(z) - \Delta Q_0(z) \ge (cM^2 - c')\,|z|^{2d-2}.$$

Choosing M large enough, we obtain $R_0(z) \ge C_3M\,|z|^{4d-4}$ by (5.1.10) again. Repeating the above argument gives $R_0(z) \ge C_pM\,|z|^{2p}$ for all sufficiently large $|\,z\,|$ for some constant $C_p > 0$. On the other hand, we will show that

$$R_0(z) \le C(1 + |z|^{4d-2}) \tag{5.1.11}$$

for all z, which will give the desired contradiction. To see this, note that for functions $u \in L^2_a(\mu_0)$, the estimate (5.1.9) gives

$$|u(z)|^2 \, e^{-Q_0(z)} \le C\delta^{-2} e^{C|\,z\,|^{2d-1}\delta} \|u\|^2_{L^2(\mu_0)}, \quad (|z| \ge 1,\ 0 < \delta < 1).$$

Taking $\delta = |\,z\,|^{1-2d}$, we obtain $|u(z)|^2 \le C|\,z\,|^{4d-2} e^{Q_0(z)} \|u\|^2_{L^2(\mu_0)}$. Since

$$L_0(z,z) = \sup\{|\,u(z)|^2 \ ;\ u \in L^2_a(\mu_0),\ \|\,u\,\|_{L^2(\mu_0)} \le 1\,\},$$

we now obtain the estimate (5.1.11). □

5.1.2 Asymptotics for $L(z,z)$

In this section, we prove part (ii) of Theorem 5.1.1.

As before, we write $Q = Q_0 + \operatorname{Re} H + Q_1$ for the canonical decomposition of Q at 0, and we write μ_0 for the measure $d\mu_0 = e^{-Q_0}\, dA$. In this section, the assumption that 0 is in the bulk of the droplet will become important.

Recall that r_n is the microscopic scale at 0 and fix a point ζ in a small neighborhood of 0 with $|\,\zeta\,| \ge r_n$. We also fix a number $\delta_0 = \delta_0(\zeta) \ge \text{const.} > 0$ with $\delta_0(\zeta) \cdot r_n/|\zeta|$ uniformly small, and a smooth function ψ with $\psi = 1$ in $D(0,\delta_0)$ and $\psi = 0$ outside $D(0, 2\delta_0)$. We define a function $\chi_\zeta = \chi_{\zeta,n}$ by

$$\chi_\zeta(\omega) = \psi((\omega - \zeta)/r_n).$$

Let $A(\eta,\omega)$ be a Hermitian-analytic function in a neighborhood of $(0,0)$, satisfying $A(\eta,\eta) = Q(\eta)$. We essentially apply the definition of the approximating kernel (denoted $L^\sharp_0$ in the preceding section) with "A_0" replaced by "nA". We write

$$\mathbf{L}^\sharp_n(\zeta,\eta) = n\, \partial_1 \bar{\partial}_2 A(\zeta,\eta) \cdot e^{\,nA(\zeta,\eta)}.$$

The corresponding "approximate projection" is defined on suitable functions u by

$$\pi^\sharp_n u(\zeta) = \langle u, \mathbf{L}^\sharp_\zeta \rangle_{L^2(\mu_n)}, \quad d\mu_n = e^{-nQ}\, dA,$$

where, for convenience, we write $\mathbf{L}^\sharp_\zeta$ instead of $\mathbf{L}^\sharp_{n,\zeta}$.

Lemma 5.1.5. *Suppose that u is holomorphic in a neighborhood of ζ and $\delta_0(\zeta) \cdot r_n/|\zeta| \le \epsilon_0$ where ϵ_0 is small enough. Then there is a constant $C = C(\epsilon_0)$ such that for $r_n \le |\,\zeta\,| \le r_n \log n$,*

$$\left| u(\zeta) - \pi^\sharp_n[\chi_\zeta u](\zeta) \right| \le C(1 + (\delta_0 r_n)^{-1}) \|\,u\,\|_{L^2(\mu_n)} e^{\,nQ(\zeta)/2}.$$

Proof. It will be sufficient to indicate how the proof of Lemma 5.1.3 is modified in the present setting. We start as earlier, by writing

$$\pi_n^\sharp[\chi_\zeta f](\zeta) = -\int \frac{u(\omega)\chi_\zeta(\omega)F(\zeta,\omega)}{\omega-\zeta}\bar\partial_\omega\left[e^{-n(A(\omega,\omega)-A(\zeta,\omega))}\right]\, dA(\omega),$$

where

$$F(\zeta,\omega) = \frac{(\omega-\zeta)\partial_1\bar\partial_2 A(\zeta,\omega)}{\bar\partial_2 A(\omega,\omega)-\bar\partial_2 A(\zeta,\omega)}.$$

Here, we may replace "A" by "A_0" to within negligible terms, for the relevant ζ and ω. More precisely, Taylor's formula gives that

$$\begin{aligned}\bar\partial_2 A(\omega,\omega)-\bar\partial_2 A(\zeta,\omega) &= \Delta Q_0(\omega)(1+O(r_n\log n))\cdot(\omega-\zeta)+O(\,(\omega-\zeta)^2\,),\\ \partial_1\bar\partial_2 A(\zeta,\omega) &= \partial_1\bar\partial_2 A_0(\zeta,\omega)(1+O(r_n\log n)),\end{aligned} \tag{5.1.12}$$

when $r_n \le |\,\zeta\,| \le r_n\log n$ and $|\omega-\zeta|\le 2\delta_0 r_n$.

From (5.1.12) and the form of F, we see (as in the proof of Lemma 5.1.3) that

$$F(\zeta,\omega) = 1+O(\zeta-\omega),\quad \bar\partial_2 F(\zeta,\omega) = O(\omega-\zeta). \tag{5.1.13}$$

We continue to write $\pi_n^\sharp u(\zeta) = u(\zeta)+\epsilon_1+\epsilon_2$ where

$$\begin{aligned}\epsilon_1 &= \int \frac{u(\omega)\cdot\bar\partial\chi_\zeta(\omega)\cdot F(\zeta,\omega)}{\omega-\zeta}e^{-n(A(\omega,\omega)-A(\zeta,\omega))}\, dA(\omega),\\ \epsilon_2 &= \int \frac{u(\omega)\cdot\chi_\zeta(\omega)\cdot\bar\partial_2 F(\zeta,\omega)}{\omega-\zeta}e^{-n(A(\omega,\omega)-A(\zeta,\omega))}\, dA(\omega).\end{aligned}$$

To estimate ϵ_1 and ϵ_2, we note that there is a positive constant c such that

$$e^{-n(Q_0(\omega)/2-\operatorname{Re}A_0(\zeta,\omega))} \le Ce^{nQ_0(\zeta)/2-cn|\zeta|^{2d-2}|\zeta-\omega|^2} \tag{5.1.14}$$

for $|\omega-\zeta|\le 2\delta_0 r_n$. See Lemma 5.1.2.

Inserting the estimates in (5.1.13) and (5.1.14), using also that $\bar\partial\chi_\zeta(w)=0$ when $|\zeta-\omega|\le\delta_0 r_n$, we find that if $|\zeta|\ge r_n$

$$\begin{aligned}|\epsilon_1|\,e^{-nQ(\zeta)/2} &\le C\delta_0^{-1}r_n^{-1}\int |u(\omega)|\,|\,\bar\partial\chi_\zeta(\omega)\,|e^{-nQ(\omega)/2}\, dA(\omega),\\ |\epsilon_2|\,e^{-nQ(\zeta)/2} &\le C\int \chi_\zeta(\omega)\,|u(\omega)|\,e^{-nQ(\omega)/2}e^{-cnr_n^{2d-2}|\zeta-\omega|^2}\, dA(\omega).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we find that

$$(|\epsilon_1| + |\epsilon_2|)e^{-nQ(\zeta)/2} \leq C(1 + \delta_0^{-1} r_n^{-1}) \| u \|_{L^2(\mu_n)}.$$

The proof is complete. □

Choosing $u(\eta) = \mathbf{k}_n(\eta, \zeta)$ where $\mathbf{k}_n$ is the Bergman kernel for the subspace $\mathcal{P}_n$ of $L^2(\mu_n)$, we obtain the following estimate, valid when $r_n \leq |\zeta| \leq r_n \log n$:

$$\left| \mathbf{k}_n(\zeta, \zeta) - \pi_n \left[\chi_\zeta \mathbf{L}_\zeta^\sharp \right] (\zeta) \right| \leq C r_n^{-1} \sqrt{\mathbf{k}_n(\zeta, \zeta)} \cdot e^{nQ(\zeta)/2}. \tag{5.1.15}$$

Here $\pi_n : L^2(\mu_n) \to \mathcal{P}_n$ is the orthogonal projection, $\pi_n u(\zeta) = \langle u, \mathbf{k}_{n,\zeta} \rangle_{L^2(\mu_n)}$.

Lemma 5.1.6. *For all ζ in the annulus $r_n \leq |\,\zeta\,| \leq \log n \cdot r_n$ and for $\delta_0(\zeta)$ with $\delta_0(\zeta) \cdot r_n$ a small enough multiple of $|\zeta|$, we have the estimate*

$$\left| \pi_n \left[\chi_\zeta \mathbf{L}_\zeta^\sharp \right] (\zeta) - n \Delta Q(\zeta)\, e^{nQ(\zeta)} \right| \leq C \sqrt{n} r_n^{-1} |\,\zeta\,|^{d-1} e^{nQ(\zeta)}.$$

Proof. Let $u_0 = \chi_\zeta \mathbf{L}_\zeta^\sharp - \pi_n \left[\chi_\zeta \mathbf{L}_\zeta^\sharp \right]$ be the norm-minimal solution in $L^2(\mu_n)$ to the problem $\bar{\partial} u_0 = \bar{\partial} f$ where $f = \chi_\zeta \mathbf{L}_\zeta^\sharp$. We claim that the problem $\bar{\partial} u = \bar{\partial} f$ has a solution u with $u - f \in \mathrm{Pol}(n)$ and

$$\| u \|_{L^2(\mu_n)} \leq C n^{-1/2} |\zeta|^{-(d-1)} \left\| \bar{\partial} \left[\chi_\zeta \mathbf{L}_\zeta^\sharp \right] \right\|_{L^2(\mu_n)}. \tag{5.1.16}$$

Let $\check{Q}$ be the "obstacle function" defined in Section 2.3.1. The obstacle function can be defined as $\check{Q} = \gamma - 2U^\sigma$ where U^σ is the logarithmic potential of the equilibrium measure and γ is a constant chosen so that $\check{Q} = Q$ on S. One has that $\check{Q}$ is harmonic outside S and its gradient is Lipschitz continuous on $\mathbb{C}$. Furthermore, $\check{Q}(\omega)$ grows like $2 \log |\omega| + O(1)$ as $\omega \to \infty$.

We use the obstacle function to form the strictly subharmonic function $\phi(\omega) = \check{Q}(\omega) + n^{-1} \log(1 + |\omega|^2)$, and we go on to define a measure μ_n' by $d\mu_n'(\omega) = e^{-n\phi(\omega)}\, dA(\omega)$. Write $\mathcal{P}_n'$ for the subspace of $L^2(\mu_n')$ of holomorphic polynomials of degree at most $n-1$, and let π_n' be the corresponding orthogonal projection. Finally, we put

$$v_0 = f - \pi_n' f.$$

Since ϕ is now strictly subharmonic, the standard Hörmander estimate can be applied. It gives

$$\| v_0 \|^2_{L^2(\mu'_n)} \le \int_{\mathbb{C}} \left|\bar{\partial} f\right|^2 \frac{e^{-n\phi}}{n\Delta\phi}\, dA.$$

Since χ_ζ is supported in the disk $D(\zeta, 2\delta_0 r_n)$, and since $\Delta\check{Q} = \Delta Q = \Delta Q_0 \cdot (1 + o(1))$ there, we see that

$$\| v_0 \|_{L^2(\mu'_n)} \le C n^{-1/2} |\zeta|^{-(d-1)} \left\| \bar{\partial} f \right\|_{L^2(\mu_n)}.$$

Next we use the estimate $n\phi \le nQ + \text{const.}$ which holds by the growth assumption on Q near infinity. This gives $\| v_0 \|_{L^2(\mu_n)} \le C \| v_0 \|_{L^2(\mu'_n)}$, and so we have shown (5.1.16) with $u = v_0$.

Since $n\phi(\omega) = (n+1)\log|\omega|^2 + O(1)$ as $\omega \to \infty$, we have that $L^2_a(\mu'_n) = \mathrm{Pol}(n)$. Hence $u = v_0$ solves, in addition to (5.1.16), the problem

$$\bar{\partial} u = \bar{\partial} f \quad \text{and} \quad u - f \in \mathrm{Pol}(n).$$

Using the form of $\bar{\partial} f = \bar{\partial}\chi_\zeta \cdot \mathbf{L}^\sharp_\zeta$ and the estimate (5.1.14), we find that for $|\omega - \zeta| \le \delta_0 r_n$,

$$\left| \bar{\partial} u(\omega) \right|^2 e^{-nQ(\omega)} \le C(n\Delta Q_0(\zeta))^2 \left| \bar{\partial}\chi_\zeta(\omega) \right|^2 e^{nQ(\zeta) - 2nc|\zeta|^{2d-2} \left|\omega - \zeta\right|^2}.$$

By the homogeneity of ΔQ_0 and the fact that $\bar{\partial}\chi_\zeta = 0$ when $|\omega - \zeta| \le \delta_0 r_n$ this gives the estimate

$$\| \bar{\partial} f \|_{L^2(\mu_n)} \le C n \left|\zeta\right|^{2d-2} e^{nQ(\zeta)/2} e^{-cn|\zeta|^{2d-2}(\delta_0 r_n)^2}.$$

Applying (5.1.16), we now get

$$\| u \|_{L^2(\mu_n)} \le C\sqrt{n} \left| \zeta \right|^{d-1} e^{nQ(\zeta)/2} e^{-cn(\delta_0 r_n)^2 |\zeta|^{2d-2}}. \tag{5.1.17}$$

We now pick a small constant δ (independent of n) and use the pointwise-L^2 estimate

$$|u(\zeta)|^2 e^{-nQ(\zeta)} \le C(r_n\delta)^{-2} e^{c' n r_n \delta |\zeta|^{2d-1}} \| u \|^2_{nQ}.$$

Choosing $\delta_0 r_n$ as a small multiple of $|\zeta|$ and then δ small enough, can now use (5.1.17) to deduce that

$$|u(\zeta)|\, e^{-nQ(\zeta)/2} \le Cr_n^{-1}\sqrt{n}|\zeta|^{d-1}e^{nQ(\zeta)/2},$$

finishing the proof. □

Proof of Theorem 5.1.1, part (ii). Fix $\epsilon > 0$ and take ζ with $r_n \le |\zeta| \le \log n \cdot r_n$. By the estimate (5.1.15) and Lemma 5.1.6 we have for all large n that

$$|\mathbf{R}_n(\zeta) - n\Delta Q_0(\zeta)| \le C_1 r_n^{-1}\sqrt{\mathbf{R}_n(\zeta)} + C_2 r_n^{-d-1}|\zeta|^{d-1},$$

for some constants C_1, C_2. Multiplying through by r_n^2 and writing $R_n(z) = r_n^2\,\mathbf{R}_n(\zeta)$ with $z = r_n^{-1}\zeta$, we get

$$|R_n(z) - \Delta Q_0(z)| \le C_1\sqrt{R_n(z)} + C_2|z|^{d-1}.$$

It follows that each limiting one point function R must satisfy

$$\left|R(z) - c(z)\,|z|^{2d-2}\right| \le C_1\sqrt{R(z)} + C_2|z|^{d-1}, \quad |z| \ge 1,$$

where $c(z) = \Delta Q_0(z/|\,z\,|) > 0$. The proof of part (i) of Theorem 5.1.1 shows that this is only possible if $R(z) = \Delta Q_0(z)(1 + O(|z|^{1-d}))$ as $z \to \infty$. □

5.2 Conical singularities

In this section, we prove the asymptotic estimate for a holomorphic limiting kernel in the case of conical singularities in Chapter 4.

Recall first the form of the n-dependent potential

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n}\log|\zeta|, \quad c > -1 \tag{5.2.1}$$

with canonical decomposition $Q = Q_0 + \operatorname{Re} H + Q_1$ at 0. We write $d\mu_n = e^{-nV_n}\,dA$. Let K be a limiting kernel of the rescaled eigenvalue system at 0 in Theorem 4.1.2 and $R(z) = K(z, z)$ be a limiting one point function. The following theorem is the main goal in this section.

Theorem 5.2.1. *There exists a constant* $\alpha = \alpha[Q_0] > 0$ *such that*

$$R(z) = \Delta Q_0(z) \cdot (1 + O(e^{-\alpha|z|^{2d}})) \quad as \quad z \to \infty. \tag{5.2.2}$$

We first start with the following lemma.

Lemma 5.2.2. *Let* $\delta = m'|\zeta|$ *where* $m' < 1$ *and put* $D_\delta = D(\zeta, \delta)$ *where we assume* $r_n \le |\zeta| \le r_n \log n$. *There are then constants* C *and* $\nu' > 0$ *such that*

$$|u(\zeta)|^2 \, e^{-nV_n(\zeta)} \le C(m'|\zeta|)^{-2} e^{\nu' m' n|\zeta|^{2d}} \|u\|^2_{L^2(\mu_n)}.$$

The constants C *and* ν' *can moreover be chosen independent of* m'.

Proof. Consider the function $F(\omega) = |u(\omega)|^2 \, e^{-nV_n(\omega)+na|\omega|^2}$, where $a > \sup_{D_\delta} \Delta Q$. Clearly, F is subharmonic in D_δ. Since $\Delta Q = (1 + o(1))\Delta Q_0$ on D_δ and Q_0 is homogeneous, we can choose a proportional to $|\zeta|^{2d-2}$. The sub-mean property gives that for some constant $c' = c'[Q_0]$,

$$\begin{aligned}|u(\zeta)|^2 \, e^{-nV_n(\zeta)} &\le C \frac{1}{\delta^2} \int_{D_\delta} |u(\omega)|^2 \, e^{-nV_n(\omega)} e^{c'n|\zeta|^{2d-2}(|\omega|^2 - |\zeta|^2)} \, dA(\omega) \\ &\le C' e^{3c'm'n|\zeta|^{2d}} \frac{1}{\delta^2} \int_{D_\delta} |u(\omega)|^2 \, e^{-nV_n(\omega)} \, dA(\omega).\end{aligned}$$

The proof of the lemma is complete. □

Fix a point ζ in the annular region $r_n \le |\zeta| \le r_n \log n$. For a small enough positive constant m with $m < 1$ we define $\tilde{\rho} = m|\zeta|/2$ and $\rho = m|\zeta|$. We also fix a smooth function ψ with $\psi = 1$ in $D(0, \tilde{\rho})$ and $\psi = 0$ outside $D(0, \rho)$. Fix a function $\chi_\zeta = \chi_{\zeta,n}$ with $\chi(\omega) = 1$ when $|\omega - \zeta| \le \tilde{\rho}$, $\chi(\omega) = 0$ when $|\omega - \zeta| \ge \rho$ and $\|\bar{\partial}\chi_\zeta\|_{L^2} \le C$.

Let $Q(\eta, \omega)$ be a Hermitian-analytic function defined in a neighborhood of the origin in $\mathbb{C}^2$, satisfying $Q(\eta, \eta) = Q(\eta)$. We write

$$V_n(\eta, \omega) = Q(\eta, \omega) - \frac{c}{n}(\log|\eta| + \log|\omega|).$$

Hence $V_n(\eta, \eta) = V_n(\eta)$, see (5.2.1).

As in the preceding section, we write

$$\mathbf{L}_n^\sharp(\zeta, \eta) = n \, \partial_1 \bar{\partial}_2 V_n(\zeta, \eta) \cdot e^{nV_n(\zeta,\eta)}.$$

The corresponding approximate projection $\pi_n^\sharp$ is defined, for suitable functions u, by

$$\pi_n^\sharp u(\zeta) = \langle u, \mathbf{L}_\zeta^\sharp \rangle_{L^2(\mu_n)}, \quad d\mu_n(\eta) = e^{-nV_n(\eta)}\, dA(\eta),$$

where we write $\mathbf{L}_\zeta^\sharp$ instead of $\mathbf{L}_{n,\zeta}^\sharp$ for convenience.

Lemma 5.2.3. *Suppose that u is holomorphic in a neighbourhood of ζ. When $r_n \le |\,\zeta\,| \le r_n \log n$, we have the estimate*

$$\left| u(\zeta) - \pi_n^\sharp[\chi_\zeta u](\zeta) \right| \le M_n(\zeta) \|\, u \,\|_{L^2(\mu_n)} e^{nV_n(\zeta)/2},$$

where

$$M_n(\zeta) = C(n^{-1/2}|\zeta|^{1-d} + |\zeta|^{-1} e^{-\nu n|\zeta|^{2d}}).$$

Here C, ν are positive constants.

Proof. We start by writing

$$\pi_n^\sharp[\chi_\zeta u](\zeta) = -\int \frac{u(\omega)\chi_\zeta(\omega)F_n(\zeta,\omega)}{\omega - \zeta} \bar\partial_\omega \left[e^{-n(V_n(\omega,\omega) - V_n(\zeta,\omega))} \right] dA(\omega),$$

where

$$F_n(\zeta,\omega) = \frac{(\omega-\zeta)\partial_1\bar\partial_2 V_n(\zeta,\omega)}{\bar\partial_2 V_n(\omega,\omega) - \bar\partial_2 V_n(\zeta,\omega)}.$$

Recall that $V_n(\zeta,\omega) = Q(\zeta,\omega) - \frac{c}{n}(\log|\zeta| + \log|\omega|)$. In the above expression for F_n it is possible to replace Q by Q_0. Indeed, Taylor's formula gives that

$$\begin{cases} \bar\partial_2 Q(\omega,\omega) - \bar\partial_2 Q(\zeta,\omega) & = \Delta Q_0(\omega)(\omega-\zeta)(1 + O(r_n \log n)) \\ & \quad + O(\,(\omega-\zeta)^2\,), \\ \partial_1\bar\partial_2 Q(\zeta,\omega) & = \partial_1\bar\partial_2 Q_0(\zeta,\omega)(1 + O(r_n\log n)), \end{cases} \tag{5.2.3}$$

when $r_n \le |\,\zeta\,| \le r_n \log n$ and $|\omega - \zeta| \le \rho$.

From (5.2.3) and the form of F_n, one checks easily that

$$F_n(\zeta,\omega) = 1 + O(\zeta - \omega), \quad \bar\partial_2 F_n(\zeta,\omega) = O(\omega - \zeta) \quad \text{as} \quad \omega \to \zeta. \tag{5.2.4}$$

We can now write $\pi_n^\sharp u(\zeta) = u(\zeta) + \epsilon_1 + \epsilon_2$ where

$$\epsilon_1 = \int \frac{u(\omega)\cdot\bar\partial\chi_\zeta(\omega)\cdot F_n(\zeta,\omega)}{\omega-\zeta} e^{n(V_n(\zeta,\omega)-V_n(\omega))}\,dA(\omega),$$
$$\epsilon_2 = \int \frac{u(\omega)\cdot\chi_\zeta(\omega)\cdot\bar\partial_2 F_n(\zeta,\omega)}{\omega-\zeta} e^{n(V_n(\zeta,\omega)-V_n(\omega))}\,dA(\omega).$$

It follows from Lemma 5.1.2 that, for $|\omega-\zeta|\le\rho$,

$$e^{n(\operatorname{Re}V_n(\zeta,\omega)-V_n(\omega)/2)} \le Ce^{nV_n(\zeta)/2-\gamma n|\zeta|^{2d-2}|\zeta-\omega|^2}, \tag{5.2.5}$$

where γ is a positive constant. Inserting the estimates in (5.2.4) and (5.2.5) and using also that $\bar\partial\chi_\zeta(w)=0$ when $|\zeta-\omega|\le\tilde\rho=m|\zeta|/2$, we find that

$$|\epsilon_1|\,e^{-nV_n(\zeta)/2} \le C|\zeta|^{-1}e^{-\nu n|\zeta|^{2d}}\int |u(\omega)|\,|\,\bar\partial\chi_\zeta(\omega)\,|e^{-nV_n(\omega)/2}\,dA(\omega),$$
$$|\epsilon_2|\,e^{-nV_n(\zeta)/2} \le C\int \chi_\zeta(\omega)\,|u(\omega)|\,e^{-nV_n(\omega)/2-\gamma n|\zeta|^{2d-2}|\zeta-\omega|^2}\,dA(\omega)$$

with a suitable $\nu>0$. Estimating the right-hand sides by means of the Cauchy-Schwarz inequality and using that $\int e^{-t|w|^2}\,dA(w)=t^{-1}$, we conclude that

$$(|\epsilon_1|+|\epsilon_2|)e^{-nV_n(\zeta)/2} \le C(|\zeta|^{-1}e^{-\nu n|\zeta|^{2d}} + n^{-1/2}|\zeta|^{1-d})\|\,u\,\|_{L^2(\mu_n)}.$$

The proof is complete. □

Choosing $u(\eta)=\mathbf{k}_n(\eta,\zeta)$ in Lemma 5.2.3, where $\mathbf{k}_n$ is the Bergman kernel for the polynomial subspace $\mathcal{P}_n$ of $L^2(\mu_n)$, one obtains after a short calculation the following estimate, valid when $r_n\le|\zeta|\le r_n\log n$:

$$\left|\mathbf{k}_n(\zeta,\zeta) - \pi_n\left[\chi_\zeta\mathbf{L}_\zeta^\sharp\right](\zeta)\right| \le M_n(\zeta)\sqrt{\mathbf{k}_n(\zeta,\zeta)}\cdot e^{nV_n(\zeta)/2}. \tag{5.2.6}$$

Here $\pi_n: L^2(\mu_n)\to\mathcal{P}_n$ is the orthogonal projection, i.e.,

$$\pi_n u(\zeta) = \langle u, \mathbf{k}_{n,\zeta}\rangle_{L^2(\mu_n)}.$$

Lemma 5.2.4. *For all ζ in the annulus $r_n\le|\zeta|\le\log n\cdot r_n$ we have the estimate*

$$\left|\pi_n\left[\chi_\zeta\mathbf{L}_\zeta^\sharp\right](\zeta) - n\Delta Q(\zeta)\,e^{nV_n(\zeta)}\right| \le N_n(\zeta)\,e^{nV_n(\zeta)}$$

where

$$N_n(\zeta) = C\sqrt{n}|\,\zeta\,|^{d-2}e^{-dn|\zeta|^{2d}}.$$

Here C, d are positive constants.

Proof. Let $u = \chi_\zeta \mathbf{L}^\sharp_\zeta - \pi_n\left[\chi_\zeta \mathbf{L}^\sharp_\zeta\right]$ be the norm-minimal solution in $L^2(\mu_n)$ to the problem $\bar\partial u = \bar\partial f$ where $f = \chi_\zeta \mathbf{L}^\sharp_\zeta$.

By the standard technique with Hörmander estimates, as in the proof of Lemma 5.1.6, we verify that there is an element $v_0 \in L^2(\mu_n)$ satisfying

$$\begin{cases} \bar\partial v_0 = \bar\partial(\chi_\zeta \mathbf{L}^\sharp_\zeta), \quad v_0 - \chi_\zeta \mathbf{L}^\sharp_\zeta \in \mathrm{Pol}(n), \\ \|\, v_0 \,\|_{L^2(\mu_n)} \le Cn^{-1/2}|\zeta|^{1-d}\left\|\,\bar\partial\left[\chi_\zeta \mathbf{L}^\sharp_\zeta\right]\,\right\|_{L^2(\mu_n)}. \end{cases}$$

The norm-minimality of u then implies

$$\|\, u \,\|_{L^2(\mu_n)} \le Cn^{-1/2}|\zeta|^{1-d}\left\|\,\bar\partial\left[\chi_\zeta \mathbf{L}^\sharp_\zeta\right]\,\right\|_{L^2(\mu_n)}. \tag{5.2.7}$$

To see that the proof in Section 5.1.2 goes through, it suffices to note that the weight V_n is subharmonic and smooth on the support of the cut-off function χ_ζ.

By Lemma 5.1.2, we have when $r_n \le |\,\zeta\,| \le r_n \cdot \log n$ and $|\omega - \zeta| \le m|\zeta|$ the estimate

$$e^{n\,\mathrm{Re}\,V_n(\zeta,\omega) - nV_n(\omega)/2} \le Ce^{nV_n(\zeta)/2\,-\gamma n|\zeta|^{2d-2}\,|\omega-\zeta|^{\,2}/2}.$$

Using the form of $\bar\partial u$, this leads to

$$\begin{aligned} |\,\bar\partial u(\omega)\,|^2 e^{-nV_n(\omega)} &= |\bar\partial\chi_\zeta(\omega)|^2 n^2 |\partial_1\bar\partial_2 Q(\zeta, w)|^2 e^{n(2\,\mathrm{Re}\,V_n(\zeta,\omega) - V_n(\omega))} \\ &\le C(n\Delta Q_0(\zeta))^2 |\,\bar\partial\chi_\zeta(\omega)\,|^2 e^{nV_n(\zeta) - \gamma(m/2)^2 n|\zeta|^{2d}}. \end{aligned}$$

By the homogeneity of ΔQ_0, this gives the estimate

$$\|\,\bar\partial(\chi_\zeta \mathbf{L}^\sharp_\zeta)\,\|_{L^2(\mu_n)} \le Cn\,|\zeta|^{\,2d-2}\,e^{-\nu n|\zeta|^{2d}}e^{nV_n(\zeta)/2},$$

with a suitable $\nu > 0$. Applying (5.2.7), we now get

$$\|\, u \,\|_{L^2(\mu_n)} \le C\sqrt{n}|\,\zeta\,|^{d-1}e^{-\nu n|\zeta|^{2d}}e^{nV_n(\zeta)/2}. \tag{5.2.8}$$

Choosing $m'\nu' \le \nu$ in Lemma 5.2.2 and applying the estimate in (5.2.8), we obtain that

$$|u(\zeta)|\, e^{-nV_n(\zeta)/2} \le C\sqrt{n}|\zeta|^{d-1}e^{nV_n(\zeta)/2}\cdot|\zeta|^{-1}e^{-\tilde{\nu}n|\zeta|^{2d}},$$

where $\tilde{\nu} = \nu - m'\nu'/2$ is a positive constant. $\square$

Proof of Theorem 5.2.1. Fix $\epsilon > 0$ and take ζ with $r_n \le |\,\zeta\,| \le \log n \cdot r_n$. By the estimate (5.2.6) and Lemma 5.2.4 we have for all large n that

$$|\mathbf{R}_n(\zeta) - n\Delta Q_0(\zeta)| \le M_n(\zeta)\sqrt{\mathbf{R}_n(\zeta)} + N_n(\zeta).$$

Multiplying through by r_n^2 and writing $R_n(z) = r_n^2\,\mathbf{R}_n(\zeta)$, $z = r_n^{-1}\zeta$, we get

$$|R_n(z) - \Delta Q_0(z)| \le r_n M_n(r_n z)\sqrt{R_n(z)} + r_n^2 N_n(r_n z).$$

A calculation shows that

$$\begin{cases} r_n M_n(r_n z) &= C(r_n^2|z|^{1-d} + |z|^{-1}e^{-\nu|z|^{2d}}),\\ r_n^2 N_n(r_n z) &= C|z|^{d-2}e^{-\tilde{\nu}|z|^{2d}}. \end{cases}$$

Letting $R = \lim R_{n_k}$, we get for $|z| \ge 1$ (assuming that $\tilde{\nu} \le \nu$)

$$|R(z) - \Delta Q_0(z)| \le (C_1|z|^{-1}\sqrt{R(z)} + C_2|z|^{d-2})e^{-\tilde{\nu}|z|^{2d}}. \tag{5.2.9}$$

Let M be a large constant, and assume that

$$|R(z) - \Delta Q_0(z)| \ge M\Delta Q_0(z)e^{-\tilde{\nu}|z|^{2d}/2} \quad \text{for} \quad |z| \ge C, \tag{5.2.10}$$

where C is large. Then (5.2.9) gives

$$R(z) \ge M' e^{\tilde{\nu}|z|^{2d}}$$

where $M' = M'(C, M)$ is a new constant. However, by the estimate of Corollary 4.2.5, we have the bound

$$R(z) \le B|z|^{4d-2} \qquad \text{for} \quad |z| \ge 1.$$

This contradiction shows that the assumption (5.2.10) must be false when

M is large enough, i.e., we have shown that

$$R(z) = \Delta Q_0(z)(1 + O(e^{-\alpha|z|^{2d}})) \quad \text{as} \quad |z| \to \infty,$$

where $\alpha = \tilde{\nu}/2$. $\square$

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국문초록

본 학위 논문에서는 랜덤 정규 행렬(random normal matrix)의 고유값들이 특이점 근방에서 이루는 확률분포를 연구한다. 랜덤 정규 행렬의 고유값들은 외부 포텐셜(external potential)이 주어져있는 볼츠만-깁스(Boltzmann-Gibbs)분포를 따른다. 외부 포텐셜이 무한대 근처에서 충분히 빠르게 증가하도록 주어지면, 행렬의 크기가 커짐에 따라 고유값들은 근사적으로 평형 측도(equilibrium measure)를 따라 분포하며 복소 평면 위의 옹골집합(compact set)에 모이게 된다.

이 옹골집합 내부에서 평형 측도의 밀도함수가 0이 되는 점을 내부 특이점(bulk singularity)이라 하며, 옹골집합 내부에서 로그 특이성을 갖는 점을 원뿔 특이점(conical singularity)이라 한다. 본 학위 논문에서는 이 두 종류의 특이점 근방에서 표준화된 고유값 분포의 극한과 그 극한의 보편성(universality)에 관해 논의한다.

주요어휘: 랜덤 정규 행렬, 내부 특이점, 원뿔 특이점, 워드 방정식, 보편성
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